常用矢量波函数的坐标变换关系*

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摘要　本文导出标准直角、圆柱和圆球矢量波函数的坐标变换关系，为实际使用提供了方便

关键词　矢量波函数；标量波函数；加法定理；坐标变换

1. 引言

为了自动、快速、准确地获取更多的电磁信息，广泛地使用了微波扫描技术．微波遥感[1]、成象[2]、多体多次散射[3,4]、逆散射[5]和天线近区场测量[6]中，经常人为地实现源点或场点以某一确定形式有规律运动．处理这一类问题，进行坐标变换并使用矢量波函数的变换关系，将会给问题求解带来很多方便．

2. 标准直角矢量波函数的变换关系

如图1所示，直角坐标系 o'-$x'y'z'$ 是由 o-xyz 经过平移（图2(a)）和旋转（图2(b)）构成的．坐标平移后，两个系统的关系是

$$\left.\begin{aligned} \hat{x} &= \hat{x}'' \\ \hat{y} &= \hat{y}'' \\ \hat{z} &= \hat{z}'' \\ \bar{R} &= \bar{R}_0 + \bar{R}'' \end{aligned}\right\} \tag{1}$$

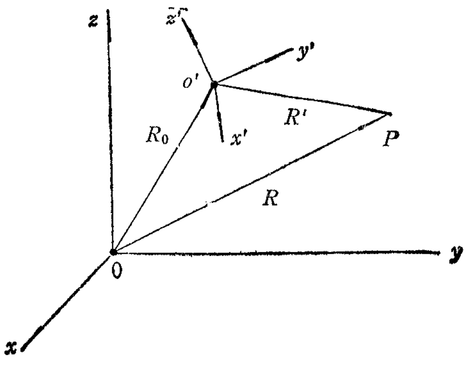

图 1

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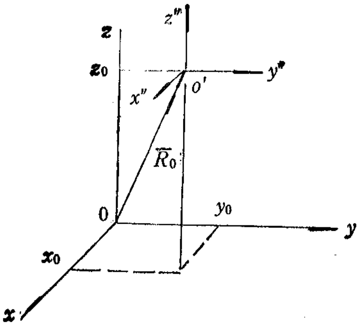
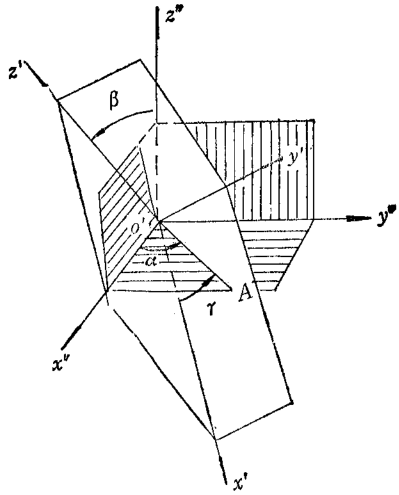


(a)



(b)

图 2

任取 $\hat{z}''$ (或 $\hat{x}''$, $\hat{y}''$) 为领示矢量，$\psi(\bar{k}, \bar{R}'') = e^{i\bar{k}\cdot\bar{R}''}$ 为生成函数，可以得到坐标系统平移后标准直角矢量波函数[7]的变换关系是

$$\begin{bmatrix} \bar{L}^{(z'')}(\bar{k},\bar{R}'') \\ \bar{M}^{(z'')}(\bar{k},\bar{R}'') \\ \bar{N}^{(z'')}(\bar{k},\bar{R}'') \end{bmatrix} = e^{-i\bar{k}\cdot\bar{R}_0} \begin{bmatrix} \bar{L}^{(z)}(\bar{k},\bar{R}) \\ \bar{M}^{(z)}(\bar{k},\bar{R}) \\ \bar{N}^{(z)}(\bar{k},\bar{R}) \end{bmatrix} \tag{2}$$

直角坐标系统旋转后，假设 oA 是面 $o'x''y''$ 与面 $o'x'y'$ 的交线，旋转 Euler 角定义为自转角 α，章动角 β 和进动角 γ，则两个系统的坐标关系是

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = [c_{ij}] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\cos\alpha\sin\gamma - \cos\beta\sin\alpha\cos\gamma & \sin\beta\sin\alpha \\ \sin\alpha\cos\gamma + \cos\beta\cos\alpha\sin\gamma & -\sin\alpha\sin\gamma + \cos\beta\cos\alpha\cos\gamma & -\sin\beta\cos\alpha \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{3a}$$

$$\begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = [c_{ij}] \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$$

和

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [d_{ij}] \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = [d_{ij}] \begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} \tag{3b}$$

其中 $[d_{ij}]=[c_{ij}]^{-1}$，选 $\hat{z}'$ 为领示矢量，$\psi(\bar{k},\bar{R}')=e^{i\bar{k}\cdot\bar{R}'}$ 为生成函数，可得坐标系统旋转后标准直角矢量波函数的变换关系是

$$\begin{aligned}
\bar{L}^{(z')}(\bar{k},\bar{R}') &= \bar{L}^{(z'')}(\bar{k},\bar{R}'') \\
\bar{M}^{(z')}(\bar{k},\bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2+k_y^2}-\frac{d_{23}k_y h}{k_x^2+k_y^2}+d_{33}\right)\bar{M}^{(z'')}(\bar{k},\bar{R}'') \\
&\quad + i\left(\frac{-d_{31}k_y k}{k_x^2+k_y^2}+\frac{d_{32}k_x k}{k_x^2+k_y^2}\right)\bar{N}^{(z'')}(\bar{k},\bar{R}'') \\
\bar{N}^{(z')}(\bar{k},\bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2+k_y^2}-\frac{d_{32}k_y h}{k_x^2+k_y^2}+d_{33}\right)\bar{N}^{(z'')}(\bar{k},\bar{R}'') \\
&\quad + i\left(-\frac{d_{31}k_y k}{k_x^2+k_y^2}+\frac{d_{32}k_x k}{k_x^2+k_y^2}\right)\bar{M}^{(z'')}(\bar{k},\bar{R}'')
\end{aligned} \tag{4}$$

式中 $k^2=k_x^2+k_y^2+h^2$. 综合(2)和(4)式最后即导出直角坐标系统经过平移和旋转以后，标准直角矢量波函数的变换关系是

$$\begin{aligned}
\bar{L}^{(z')}(\bar{k},\bar{R}') &= e^{-i\bar{k}\cdot\bar{R}_0}\bar{L}^{(z)}(\bar{k},\bar{R}) \\
\bar{M}^{(z')}(\bar{k},\bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2+k_y^2}-\frac{d_{32}k_y h}{k_x^2+k_y^2}+d_{33}\right)e^{-i\bar{k}\cdot\bar{R}_0}\bar{M}^{(z)}(\bar{k},\bar{R}) \\
&\quad + i\left(-\frac{d_{31}k_y k}{k_x^2+k_y^2}+\frac{d_{32}k_x k}{k_x^2+k_y^2}\right)e^{-i\bar{k}\cdot\bar{R}_0}\bar{N}^{(z)}(\bar{k},\bar{R}) \\
\bar{N}^{(z')}(\bar{k},\bar{R}') &= \left(-\frac{d_{31}k_x h}{k_x^2+k_y^2}-\frac{d_{32}k_y h}{k_x^2+k_y^2}+d_{33}\right)e^{-i\bar{k}\cdot\bar{R}_0}\bar{N}^{(z)}(\bar{k},\bar{R}) \\
&\quad + i\left(-\frac{d_{31}k_y k}{k_x^2+k_y^2}+\frac{d_{32}k_x k}{k_x^2+k_y^2}\right)e^{-i\bar{k}\cdot\bar{R}_0}\bar{M}^{(z)}(\bar{k},\bar{R})
\end{aligned} \tag{5}$$

3. 标准圆柱矢量波函数的变换关系

如图 3 所示，圆柱坐标系统 $o\text{-}r\varphi z$ 变换到 $o'\text{-}r'\varphi'z'$ 有下列关系，即

$$\begin{aligned}
r &= r_0\cos(\varphi-\varphi_0)+r'\cos(\varphi'-\varphi) \\
\varphi &= \varphi'-\eta \\
z &= z_0+z'
\end{aligned} \tag{6}$$

由此得到柱谐函数 $J_n(\lambda r')e^{in\varphi'}$ 的表示式是[8]

$$J_n(\lambda r')e^{in\varphi'} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda r'\cos\theta+in\left(\theta+\varphi'-\frac{\pi}{2}\right)}\,d\theta$$

$$= \sum_{m=-\infty}^{\infty} J_m(\lambda r_0)J_{n+m}(\lambda r)e^{in\varphi+im(\varphi-\varphi_0)} \tag{7a}$$

或

$$J_n(\lambda r')e^{in\eta} = \sum_{m=-\infty}^{\infty} J_m(\lambda r_0)J_{n+m}(\lambda r)e^{im(\varphi-\varphi_0)} \tag{7b}$$

同理，当 $|r|>|r_0\cos(\varphi-\varphi_0)|$ 时，有

$$H_n^{(1)}(\lambda r')e^{in\eta} = \sum_{m=-\infty}^{\infty} J_m(\lambda r_0)H_{n+m}^{(1)}(\lambda r)e^{im(\varphi-\varphi_0)} \tag{8a}$$

当 $|r|<|r_0\cos(\varphi-\varphi_0)|$ 时，有

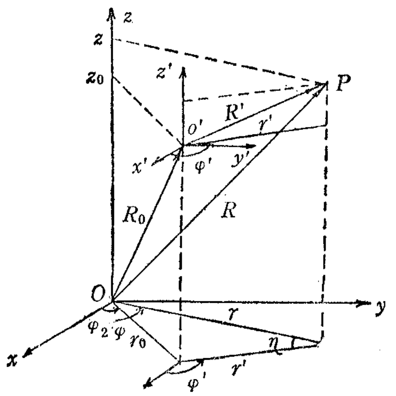


图 3

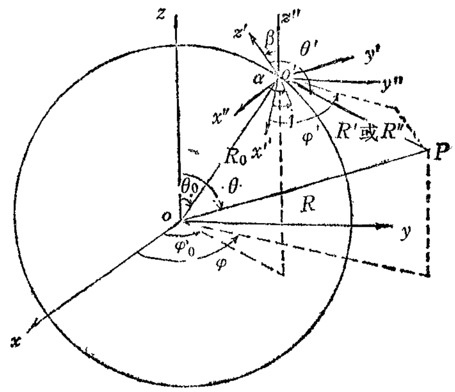


图 4

$$H_n^{(1)}(\lambda r')e^{in\eta}=\sum_{m=-\infty}^{\infty}H_m^{(1)}(\lambda r_0)J_{n+m}(\lambda r)^{im(\varphi-\varphi_0)} \tag{8b}$$

利用标量柱谐函数加法定理(7)和(8)式，可以导得标准圆柱矢量波函数的坐标关系是

$$\begin{bmatrix}\bar{L}_n^{(1)}(\bar{k},\bar{R}')\\ \bar{M}_n^{(1)}(\bar{k},\bar{R}')\\ \bar{N}_n^{(1)}(\bar{k},\bar{R}')\end{bmatrix}=\sum_{m=-\infty}^{\infty}J_m(\lambda r_0)e^{-i(m\varphi_0+hz_0)}\begin{bmatrix}\bar{L}_{n+m}^{(1)}(\bar{k},\bar{R})\\ \bar{M}_{n+m}^{(1)}(\bar{k},\bar{R})\\ \bar{N}_{n+m}^{(1)}(\bar{k},\bar{R})\end{bmatrix} \tag{9a}$$

当 $|r|>|r_0\cos(\varphi-\varphi_0)|$ 时

$$\begin{bmatrix}\bar{L}_n^{(3)}(\bar{k},\bar{R}')\\ \bar{M}_n^{(3)}(\bar{k},\bar{R}')\\ \bar{N}_n^{(3)}(\bar{k},\bar{R}')\end{bmatrix}=\sum_{m=-\infty}^{\infty}J_m(\lambda r_0)e^{-i(m\varphi_0+hz_0)}\begin{bmatrix}\bar{L}_{n+m}^{(3)}(\bar{k},\bar{R})\\ \bar{M}_{n+m}^{(3)}(\bar{k},\bar{R})\\ \bar{N}_{n+m}^{(3)}(\bar{k},\bar{R})\end{bmatrix} \tag{9b}$$

当 $|r|<|r_0\cos(\varphi-\varphi_0)|$ 时

$$\begin{bmatrix}\bar{L}_n^{(3)}(\bar{k},\bar{R}')\\ \bar{M}_n^{(3)}(\bar{k},\bar{R}')\\ \bar{N}_n^{(3)}(\bar{k},\bar{R}')\end{bmatrix}=\sum_{m=-\infty}^{\infty}H_m^{(1)}(\lambda r_0)e^{-(m\varphi_0+hz_0)}\begin{bmatrix}\bar{L}_{n+m}^{(1)}(\bar{k},\bar{R})\\ \bar{M}_{n+m}^{(1)}(\bar{k},\bar{R})\\ \bar{N}_{n+m}^{(1)}(\bar{k},\bar{R})\end{bmatrix} \tag{9c}$$

4. 标准圆球矢量波函数的变换关系

如图 4 所示，圆球坐标系统 o-$R\theta\varphi$ 经过平移 $\bar{R}_0$ 后成 o'-$R''\theta''\varphi''$，再旋转 Euler 角，成系统 o'-$R'\theta'\varphi'$。S. Stein[9] 曾导出平移后圆球标量波函数的加法定理是

$$z_n(kR)P_n^m(\cos\theta)e^{im\varphi}=\begin{cases}\displaystyle\sum_{p=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}i^{\nu+p-n}(2\nu+1)a(m,n|-\mu\varphi|p)\\ \quad\times j_\nu(kR'')z_p(kR_0)P_\nu^\mu(\cos\theta'')P_p^{m-\mu}(\cos\theta_0)\\ \quad\times e^{i(m-\mu)\varphi_0}e^{i\mu\varphi''}, \qquad R''\leqslant R_0\\ \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}i^{\nu+p-n}(2\nu+1)a(m,n|-\mu,\nu|p)\\ \quad\times j_\nu(kR_0)z_p(kR'')P_\nu^\mu(\cos\theta_0)P_p^{m-\mu}(\cos\theta'')\\ \quad\times e^{i(m-\mu)\varphi''}e^{i\mu\varphi_0}, \qquad R''\geqslant R_0\end{cases} \tag{10}$$

其中下标 $p=\nu+n,\ \nu+n-2,\ \nu+n-4,\cdots$，但不低于 $|\nu-n|$，即 $\nu+n\geqslant p\geqslant|\nu-n|$。系数 $a(m,n|-\mu,\nu|p)$ 由展开式

$$P_n^m(\cos\theta)P_\nu^\mu(\cos\theta)=\sum_p a(m,n|\mu,\nu|p)P_p^{m+\mu}(\cos\theta) \tag{11}$$

定义,可以写成

$$a(m,n|\mu,\nu|p)=(-1)^{m+\mu}(2p+1)\left[\frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!}\right]^{1/2}\times\begin{bmatrix} n & \nu & p \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} n & \nu & p \\ m & \mu & -m-\mu\end{bmatrix} \tag{12}$$

式中 $\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{bmatrix}$ 是 Wigner 3-j 符号[10]. 经过复杂地运算[11],无论 $R''\leqslant R_0$ 或 $R''\geqslant R_0$, 式(10)可以改写成

$$z_n(kR)P_n^m(\cos\theta)e^{im\varphi}=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}A(\mu,\nu)z_\nu(kR'')P_\nu^\mu(\cos\theta'')e^{i\mu\varphi''} \tag{13}$$

其中

$$A(\mu,\nu)=(-1)^\mu i^{\nu-n}(2\nu+1)\sum_p i^p a(m,n|-\mu,\nu|p)z_p(kR_0)P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0} \tag{14}$$

由此,坐标系统平移后,标准圆球矢量波函数的变换关系是

$$\begin{aligned}\bar{K}_{mn}(\bar{k},\bar{R})&=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}A(\mu,\nu)\bar{L}_{\mu\nu}(\bar{k},\bar{R}'')\\ \bar{M}_{mn}(\bar{k},\bar{R})&=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}[A_{\mu\nu}^{mn}\bar{M}_{\mu\nu}(\bar{k},\bar{R}'')+B_{\mu\nu}^{mn}\bar{N}_{\mu\nu}(\bar{k},\bar{R}'')]\\ \bar{N}_{mn}(\bar{k},\bar{R})&=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}[A_{\mu\nu}^{mn}\bar{N}_{\mu\nu}(\bar{k},\bar{R}'')+B_{\mu\nu}^{mn}\bar{M}_{\mu\nu}(\bar{k},\bar{R}'')]\end{aligned} \tag{15}$$

其中[12]

$$\begin{aligned}A_{\mu\nu}^{mn}&=(-1)^\mu\sum_p a(m,n|-\mu,\nu|p)a(n,\nu,p)z_p(kR_0)P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0}\\ B_{\mu\nu}^{mn}&=(-1)^\mu\sum_p a(m,n|-\mu,\nu|p,p-1)b(n,\nu,p)z_p(kR_0)P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0}\\ a(n,\nu,p)&=i^{\nu+p-n}[2\nu(\nu+1)(2\nu+1)+(\nu+1)(n-\nu+p+1)(n+\nu-p)\\ b(n,\nu,p)&=i^{\nu+p-n}[(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)]^{1/2}\frac{2\nu+1}{2\nu(\nu+1)}\end{aligned} \tag{16}$$

A. R. Edmonds[10] 用转动群表示证明了标量圆球波函数的加法定理是

$$z_n(kR'')P_n^m(\cos\theta'')e^{im\varphi''}=\sum_{\mu=-n}^{n}\beta(m,\mu,n)z_n(kR')P_n^\mu(\cos\theta')e^{i\mu\varphi'} \tag{17}$$

式中

$$\begin{aligned}\beta(m,\mu,n)&=(-1)^{\mu+m}\left[\frac{(n+m)!(n-\mu)!}{(n-m)!(n+\mu)!}\right]^{\frac{1}{2}}D_{m\mu}^{(n)}(\alpha,\beta,\gamma)\\ D_{m\mu}^{(n)}(\alpha,\beta,\gamma)&=e^{i\mu\alpha}d_{m\mu}^{(n)}(\beta)e^{im\gamma}\\ d_{m\mu}^{(n)}(\beta)&=\left[\frac{(n+\mu)!(n-\mu)!}{(n+m)!(n-m)!}\right]^{1/2}\sum_\sigma\binom{n+m}{n-\mu-\sigma}\binom{n-m}{\sigma}(-1)^{n-\mu-\sigma}\\ &\quad\times\left(\cos\frac{\beta}{2}\right)^{2\sigma+\mu+m}\left(\sin\frac{\beta}{2}\right)^{2n-2\sigma-\mu-m}\end{aligned} \tag{18}$$

由此得到坐标旋转后标准圆球矢量波函数的变换关系是

$$\bar{L}_{mn}(\bar{k}, \bar{R}'') = \sum_{\mu=-n}^{n} \beta(m, \mu, n)\bar{L}_{\mu n}(\bar{k}, \bar{R}')$$

$$\bar{M}_{mn}(\bar{k}, \bar{R}'') = \sum_{\mu=-n}^{n} \beta(m, \mu, n)\bar{M}_{\mu n}(\bar{k}, \bar{R}') \tag{19}$$

$$\bar{N}_{mn}(\bar{k}, \bar{R}'') = \sum_{\mu=-n}^{n} \beta(m, \mu, n)\bar{N}_{\mu n}(\bar{k}, \bar{R}')$$

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TRANSFORMATION RELATIONS OF NORMAL VECTOR WAVE FUNCTIONS

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Abstract The transformation relations of normal rectangular, cylindrical and spherical vector wave functions are derived. With these relations, it will be convenient for engineering.

Key words Vector wave function; Scalar wave function; Addition theorem; Coordinate transformation