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极大平面图的结构与着色理论

(3)纯树着色与唯一 4-色极大平面图猜想

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摘 要：一个极大平面图若是从 K_4 出发，不断地在三角面上嵌入 3 度顶点得到的，则称此极大平面图为递归极大平面图。唯一 4-色极大平面图猜想是指：一个平面图是唯一 4-可着色的当且仅当它是递归极大平面图。此猜想已有 43 年历史，是图着色理论中继四色猜想之后另一个著名的未解猜想。为此，该文相继深入研究了哑铃极大平面图与递归极大平面图的结构与特性，结合该系列文章(2)的扩缩运算，给出了证明唯一 4-色极大平面图猜想的一种思路。

关键词：唯一 4-色极大平面图猜想；纯树着色猜想；哑铃极大平面图；递归极大平面图
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Theory on Structure and Coloring of Maximal Planar Graphs

(3) Purely Tree-colorable and Uniquely 4-colorable Maximal Planar Graph Conjectures

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Abstract: A maximal planar graph is called the recursive maximal planar graph if it can be obtained from K_4 by embedding a 3-degree vertex in some triangular face continuously. The uniquely 4-colorable maximal planar graph conjecture states that a planar graph is uniquely 4-colorable if and only if it is a recursive maximal planar graph. This conjecture, which has 43 years of history, is a very influential conjecture in graph coloring theory after the Four-Color Conjecture. In this paper, the structures and properties of dumbbell maximal planar graphs and recursive maximal planar graphs are studied, and an idea of proving the uniquely 4-colorable maximal planar graph conjecture is proposed based on the extending-contracting operation proposed in this series of article (2).

Key words: Uniquely 4-colorable maximal planar graph conjecture; Purely tree-colorable planar graph conjecture; Dumbbell maximal planar graphs; Recursive maximal planar graphs

1 引言

图论诞生于1736年EULER研究的哥尼斯堡七桥问题，以及 1847 年 KIRCHHOFF 研究的电网络问题。图论在最近 70 年来得以迅速发展，其主要原因有两个：一个是受到电子计算机发展的影响；另一个更重要的原因是受四色猜想的影响。对四色猜想的研究推动了整个图论学科的发展，开创了图论的许多领域，诸如拓扑图论、最大独立集与最大团理论、顶点与边覆盖理论、色多项式理论、Tutte-多项式理论、因子理论、整数流理论，特别是图着色理论等。

在图着色领域，除著名的四色猜想外，相继出现了其它不少著名的猜想，本文所讨论的“**唯一 4-色极大平面图猜想**”就是其中之一。唯一 4-色极大平面图猜想源于1973年GREENWELL与KRONK所提出的猜想[1]，距今已有 43 年，此猜想本质上也与四色猜想有内在联系，至今尚未解决。

图 的 唯 一 着 色 概 念 是 由 GLEASON 和

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CARTWRIGHT[2], CARTWRIGHT 和 HARARY[3]相继提出的。CARTWRIGHT 和 HARARY 给出了一些判定标号图是唯一可着色图的充分条件。其后，许多学者在此领域做了大量工作，诸如 HARARY, HEDETNIEMI 和 ROBINSON[4]研究了唯一 k-可着色图的连通性问题、边数的取值问题等；对于任意 $k \ge 3$，不含 K_3 的唯一 k-色图是否存在这个问题，众多学者展开了研究。如 NESTRIL[5,6]研究了临界-唯一可着色图的性质，并证明了存在不含三角形的唯一 k-可着色图；1974 年，GREENWELL 和 LOVASZ[7]证明了存在不含短奇圈的唯一 k-可着色图；1975 年，MULLER[8]解决了此问题的一般情形(也见文献[9])，即对任意的正整数 $k \ge 3$ 和 t，存在围长大于 t 的唯一 k-可着色图，其中 MULLER 采用的也是构造的方法；MULLER[8,9], AKSIONOV[10], MELNIKOV 与 STEINBERG[11]研究了边-临界唯一可着色图性质；WANG 和 ARTZY[12]得到了“当 $k \ge 3$，如果存在一个不含 K_3 的唯一 k-可着色图 G，那么图 G 的边数严格大于 k^2+k-1”；OSTERWEIL[13]给出 6-团环构造一类唯一 3-可着色图的方法；BOLLOBAS 和 SAUER[14]证明了“对任意 $k \ge 2$ 和 $g \ge 3$，总存在一个围长至少为 g 的唯一 k-可着色图”,其中 g 是给定图的围长,并证明了“对任意 $k \ge 3$ 和 n，总存在一个阶数至少为 n 的临界-唯一 k-色图”；DMITRIEV[15]推广了 BOLLOBAS 在文献[16]中的结果；XU[17]证明了“如果 G 是一个顶点数为 n，边数为 m 的唯一 k-可着色图，则 $m \ge (k-1)n-(1/2)k(k-1)$，且该界是最好可能的”，并猜想“如果 G 是一个顶点数为 n，边数为 $(k-1)n-(1/2)k(k-1)$ 的唯一 k-可着色图，则 G 含子图 K_k”。同时，CHAO 和 CHEN[18]证明了“对每个正整数 $n \ge 12$，存在一个不含三角形的 n-阶唯一 3-可着色图”; AKBARI, MIRROKNI 和 SADJAD[19]证明了“存在阶数为 24 不含三角形的唯一 3-可着色图且边数 $SH(G)=45$，其中 $SH(G)=(k-1)n-(1/2)k(k-1)$”，该结果否定了 XU[17]的猜想。

在边唯一可着色图方面，1973 年，GREENWELL 与 KRONK 首先研究了唯一边-可着色图[1]。他们提出了下述猜想：

猜想 1　若 G 是一个唯一 3-边可着色立方图，则 G 是平面图且含一个三角形。

1975 年，FIORINI[20]独立研究了边唯一可着色问题，并获得一些与 GREENWELL 及 KRONK[1]类似结果。其后，不少学者研究了唯一边可着色问题，如 THOMASON[21,22], FIORINI 与 WILSON[23], ZHANG[24], GOLDWASSER 与 ZHANG[25,26], KRIESSEL[27]等。

1977 年，FIORINI 和 WILSON[23]，及 FISK[28]分别独立提出下述猜想：

猜想 2　每个至少有 4 个顶点的唯一 3-边可着色立方平面图含一个三角形。

这个猜想是在猜想 1 的基础上进一步提出来的。FOWLER[29]也对此猜想进行了一定的研究。此猜想至今未被解决。

在唯一可着色平面图方面，1969 年，CHARTRAND 和 GELLER[30]开始研究唯一可着色平面图。他们证明了至少有 4 个顶点的唯一 3-可着色平面图至少含两个三角形，**唯一 4-可着色平面图是极大平面图**，不存在唯一 5-可着色平面图。

唯一 3-色平面图的充分必要条件是什么？这个问题至今尚未解决，但关于唯一 3-色平面图的一些基本特性已有很多研究。1977 年，AKSIONOV[31]证明了阶数 ≥ 6 的唯一 3-色平面图至少包含 3 个三角形，并详细刻画了恰含 3 个三角形的唯一 3-色平面图的结构特征。同年，MELNIKOV 和 STEINGERG[11]研究了边临界唯一 3-色平面图，并提出如下问题：找出 n-阶边临界唯一 3-可着色平面图边数的精确上界 size(n)。2013 年，MATSUMOTO[32]证明了 $\text{size}(n) \le (8/3)n-17/3$；最近，LI 等人[33,34]证明了 $\text{size}(n) \le (5/2)n-6$，其中 $n \ge 6$，并证明了包含至多 4 个三角形的唯一 3-色平面图中存在相邻三角形。

哪些极大平面图是唯一 4-可着色的？换言之，唯一 4-可着色平面图的基本特征是什么？这个问题自然是研究唯一 4-可着色平面图的主要内容。围绕此问题，许多学者从不同方面展开了研究[24,35−37]。

其实，FISK 在文献[28]中，也提出了与猜想 2 等价的猜想 3。

猜想 3　一个平面图 G 是唯一 4-可着色的充分必要条件是 G 为递归极大平面图。

猜想 3 与猜想 2 的等价性是容易证明的，且是猜想 1 的特殊情况。1998 年，BOHME, STIEBITZ 和 VOIGT[35]证明了猜想 3 的最小反例图是 5-连通的。我们把猜想 3 称为**唯一 4-色极大平面图猜想**。

本文所言之图皆指有限简单无向图。对于给定图 G，分别用 $V(G)$，$E(G)$，$d_G(v)$ 和 $N_G(v)$ 来表示图 G 的顶点集，边集，顶点 v 的度数和顶点 v 的**邻域**(即与顶点 v 相邻的所有顶点构成的集合)，可分别简记为 V，E，$d(v)$，$N(v)$。图 G 的**阶**是指 $V(G)$ 中元素的个数 $|V(G)|$。若 $V' \subseteq V$，$E' \subseteq E$，且 E' 中每条边的两个端点均在 V' 中，则称图 $H=(V',E')$ 是图 G 的一个**子图**。在子图 H 中，如果对于

$\forall u,v \in V(H)$，u,v 在 G 中相邻当且仅当它们在图 H 中相邻，则称 H 为 G 的一个由 V' **导出的子图**，记为 $G[V']$。对于点不交的两个图 G，H，若将图 G 中的每个顶点与图 H 中的每个顶点相连边，则得到的新图称为图 G 与图 H 的**联图**，记为 $G \vee H$。用 K_n 表示 n-阶完全图。K_1 与 n 阶圈 C_n 的联图 $C_n \vee K_1$ 称作**轮图**或 n-**轮**，记作 W_n，其中 C_n 称为 W_n 的**轮圈**；K_1 的顶点称为该轮的**轮心**。若 $V(K_1)=\{x\}$，则有时也把 W_n 的圈 C_n 用 C^x 来表示。

图 G 的一个**顶点着色** f 是指对图 G 中的每个顶点分配一种颜色，且满足 G 中每条边的两个端点分配不同的颜色。图 G 的一个 k-**顶点着色**，简称为 k-**着色**，是指从图 G 的顶点集 V 到颜色集 $C(k)=\{1,2,\cdots,k\}$ 的一个映射 f，满足对任意的 $xy \in E(G)$，有 $f(x) \neq f(y)$。如果在 G 中存在一个 k-顶点着色，则称图 G 是 k-**可着色的**。图 G 的**色数**，记作 $\chi(G)$，是指满足图 G 为 k-顶点可着色的最小数值 k。若 $\chi(G)=k$，则称 G 是 k-**色图**。图 G 的每一个 k-顶点着色 f 对应于顶点集 V 的一个划分 $\{V_1,V_2,\cdots,V_k\}$，也记作 $f=(V_1,V_2,\cdots,V_k)$，V_i 为色组，表示分配到颜色 i 的所有顶点构成的集合，即

$$V(G)=\bigcup_{i=1}^{k} V_i, V_i \neq \phi, V_i \cap V_j=\phi, i \neq j, i,j=1,2,\cdots,k$$

其中，$V_i\,(i=1,2,\cdots,k)$ 是 G 的独立集。图 G 中所有不同的 k-着色所构成的集合用 $C_k(G)$ 表示。用 $C_k^0(G)$ 表示 G 的所有由 k 个色组构成的划分的集，简称为图 G 的 k-**色组划分集**。

一个 k-色图 G 称作是**唯一 k-可着色的**，如果 G 的所有 k-着色都把 G 的顶点集 V 划分成唯一的一组(k 个)独立集。

类似地，可给出边着色的定义。图 G 称为**唯一 k-边着色的**，如果图 G 的所有 k-边着色都把图 G 的边集 E 唯一地划分成一组(k 个)匹配子集。

对于一个平面图，如果只要任何两个不相邻的顶点之间再加一条边，其平面性一定被破坏，则称该平面图为**极大平面图**。若一个平面图的每个面（包括无穷面）都由 3 条边组成，则称该平面图为**三角剖分图**。易证，极大平面图和三角剖分图是等价的。

设 G 是一个极大平面图，$\mathrm{Aut}(G)$ 是它的**自同构群**，H，H' 是 G 的两个同构子图。若 $\exists \sigma \in \mathrm{Aut}(G)$，使得 $\sigma(H)=H'$，则称 H 与 H' 是**等同的**。

文中未给出的概念与符号可查看文献[38,39]。

2 树着色与圈着色

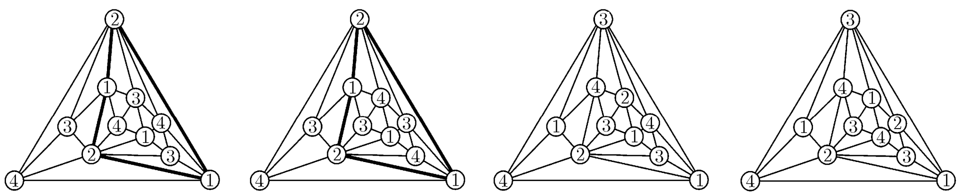
设 G 是一个 4-色极大平面图，颜色集 $C(4)=\{1,2,3,4\}$，$f \in C_4^0(G)$。若 G 中有一长度为 $2m$ 的偶圈 C_{2m}，$V(C_{2m})=\{v_1,v_2,\cdots,v_{2m}\}$，使得 $\{f(v_1), f(v_2),\cdots,f(v_{2m})\}$ 中只含有 2 种颜色，则称 C_{2m} 是 f 的一个 **2-色圈**，也称 f **含有 2-色圈**，并称 f 为**圈着色**。若 C_{2m} 上所含颜色为 i 和 t，则 C_{2m} 亦可记作 it-圈。否则，若 f 不含 2-色圈，则称 f 为图 G 的**树着色**。如图 1 中所示图的 4-着色，f_1 与 f_2 为圈着色，f_3 与 f_4 为树着色。在圈着色与树着色分类的基础上，相应地，可将 G 分为 3 种类型：**纯树着色型**，即 $C_4^0(G)$ 每个着色均为树着色；**纯圈着色型**，即 $C_4^0(G)$ 每个着色均为圈着色；**混合着色型**，即 $C_4^0(G)$ 中既含树着色，又含圈着色。如图 1 中所示的图属于混合着色型。

一个极大平面图 G 称为**可圈着色的**，如果 $\exists f \in C_4^0(G)$，f 是圈着色；G 称为**可树着色的**，如果 $\exists f \in C_4^0(G)$，f 是树着色。对阶数为7~11的所有最小度 ≥ 4 的非可分极大平面图的4-着色数目进行统计，发现：树着色数约占2%，即圈着色数约占98%。目前在可树着色极大平面图结构与特性方面的研究较少。ZHU等人[40]证明了最小度 ≥ 4 的可树着色极大平面图 G 包含至少4个度数为奇数的顶点，并且，当 G 恰含4个度数为奇数的顶点时，G 中这4个顶点的导出子图不含三角形且不是爪图。

7 至 11 阶最小度 ≥ 4 的极大平面图共有 54 个，其中纯树着色型的图只有 1 个(为方便，我们在下一节给出其图示)，称为 **9-阶哑铃极大平面图**，或**基本哑铃极大平面图**[41]，记作 J^9。

3 纯树着色极大平面图

由第2节知，极大平面图可分为纯树着色型，纯圈着色型与混合着色型3种。而刻画这3种类型极大平面图的结构，给出相应的充分必要条件是很困难的问题。如若刻画出纯树着色型的结构特征，



(a) f_1，圈着色，含12-圈 (b) f_2，圈着色，12-圈 (c) f_3，树着色 (d) f_4，树着色

图 1 一个 11-阶 4-色极大平面图的全部 4 种着色

自然也就证明了已有 43 年的唯一 4-色极大平面图猜想(猜想 3)。故从本文开始，我们将逐渐对这 3 种类型极大平面图展开研究。本节重点针对纯树着色型展开研究。

3.1 最小度为 5 的纯树着色极大平面图猜想

文献[41]中已指出：正二十面体是一个最小度为 5 的纯树着色极大平面图，它共有 10 种不同的树着色，如图 2 所示。

图 G 称为**顶点可迁的**，若对 G 中任意两个顶点 u, v，$\exists\sigma\in\mathrm{Aut}(G)$，使 $\sigma(u)=v$。易证，正二十面体是**顶点可迁的**。因此，其中所含的外圈长度为 6，且内点数为 2 的多米诺构形必是唯一的，故图 2 的第 1 个图中粗线所示的多米诺构形可作为其代表。

猜想 4　最小度为 5 的极大平面图是纯树着色的当且仅当它是正二十面体。

3.2 哑铃极大平面图

3.2.1 哑铃变换　哑铃变换的对象图是一个全封哑铃，即为一个 4-轮，如图 3 所示。所谓**哑铃变换**，是指按照如下步骤，将一个全封哑铃变成一个如图 3(a)，或图 3(b)中最右边所示构形的过程：

步骤 1　将轮心划开，横划或竖划，如图 3(a)，或图 3(b)中的第 2 个图示；

步骤 2　伸展开成如图 3(a)，或图 3(b)中所示的第 3 个图；

步骤 3　在 6-圈内添加 2-长路，并按横划开与竖划开，令 2-长路与 6-圈连接边构成如图 3 中的最右边的构形。

上述步骤的逆运算称为**哑铃收缩变换**，并把图

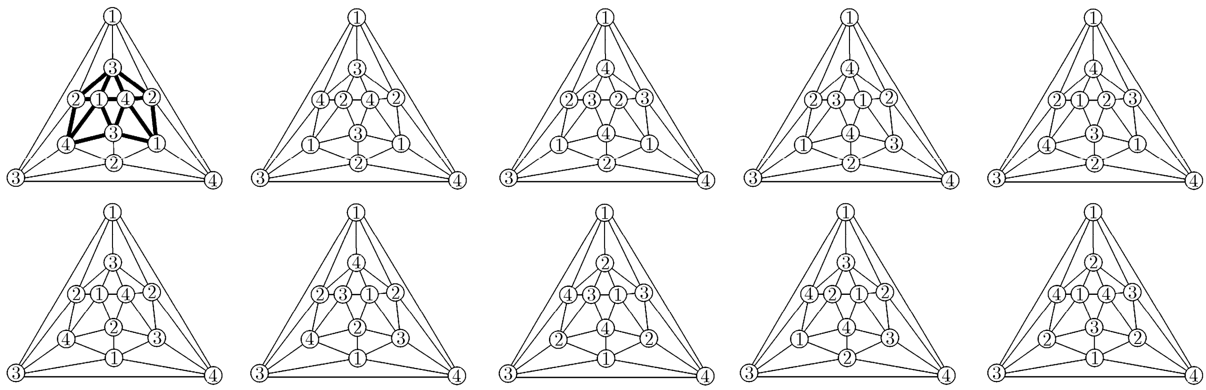


图 2 正二十面体及它的全部 10 种 4-着色

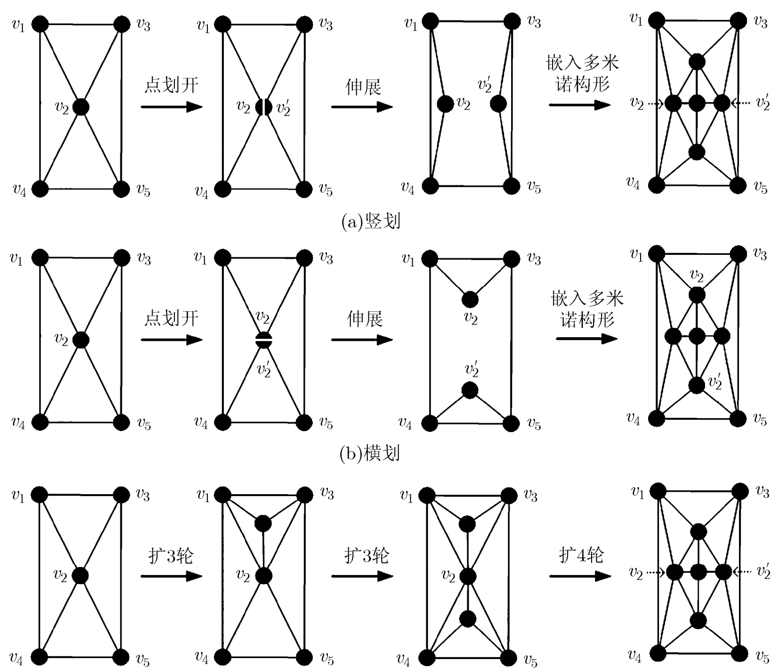


图 3 哑铃变换对象图及过程示意图

3(a)中最右边的构形称为**哑铃收缩变换对象图**。哑铃变换实际上是扩 334-轮运算，可参见图 3(c)，也可参见文献[39]。

3.2.2 哑铃极大平面图的构造 设 J^{4k+1} ($k \ge 3$)是一个最小度为 4 的 $(4k+1)$ -阶极大平面图，且含哑铃收缩变换对象子图。若对该对象子图实施哑铃收缩变换，所得之图若仍含有哑铃收缩变换对象子图，再对其实施哑铃收缩变换，按此步骤连续实施哑铃收缩变换，当最终所得之图为 J^9 时，则称 J^{4k+1} 为**哑铃极大平面图**。

由此定义可知，对一个哑铃极大平面图实施哑铃变换后所得之图必为哑铃极大平面图。

下面，基于 $(4k+1)$ - 阶哑铃极大平面图 J^{4k+1} ($k \ge 2$)，给出构造 $(4k+5)$-阶哑铃极大平面图的方法步骤：

步骤 1 找出 J^{4k+1} 中所有不等同的全封哑铃，即寻找不等同的 4-轮。如图 4(a), 4(b), 4(c), 4(d)中所示分别为 9-阶，13-阶，17-阶，17-阶哑铃极大平面图，它们中均含 3 个 4-轮，但不等同的 4-轮分别为 1, 2, 2, 2 个；

步骤 2 对 J^{4k+1} 中的每个不等同的 4-轮实施哑铃变换，得到 $(4k+5)$-阶哑铃极大平面图。如图 4(a)所示的基本哑铃极大平面图 J^9，对粗线所示的全封哑铃施行哑铃变换得 J^{13}，如图 4(b)示。J^{13} 中仍含有 3 个全封哑铃，易证，有两个是等同的，分别对其施行哑铃变换，得到如图 4(c), 4(d)所示的两个 17-阶的哑铃极大平面图。

3.2.3 哑铃极大平面图的性质 下面，我们进一步讨论哑铃极大平面图的一些性质。

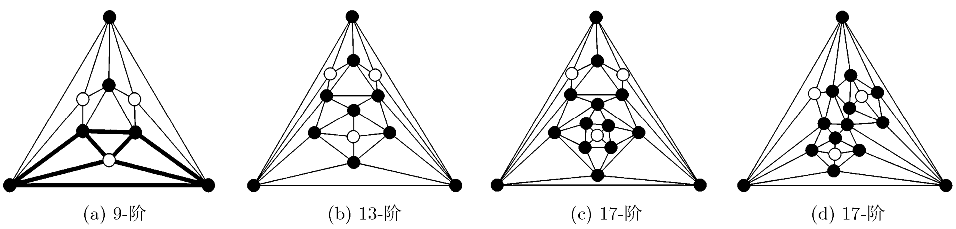
定理 1 (1)任一哑铃极大平面图恰有 3 个 4-度顶点；(2)每个哑铃极大平面图的阶数为 $4k+1$，其中 $k \ge 2$；(3)每个哑铃极大平面图均为纯树着色型，且每个 $(4k+1)$ -阶哑铃极大平面图恰有 2^{k-1} 种不同的着色。

证明 (1)与(2)的证明：注意到 J^9 有 3 个等同的 4-度顶点，因此，由哑铃变换的定义知，J^9 经过哑铃变换后得到的 13-阶哑铃极大平面图恰有 3 个 4-度顶点，逐次类推，易证对于 $\forall k \ge 2$，J^{4k+1} 恰有 3 个 4-度顶点，且每个哑铃极大平面图的阶数均为 4 的倍数余 1，即为 $4k+1$；

(3)用数学归纳法来证明：9-阶哑铃极大平面图时结论成立，13-阶的哑铃极大平面图共有 4 种着色，且均为纯树着色，如图 5 所示。故结论成立。

假设当 $k \ge 3$ 时结论成立，即 J^{4k+1} 为纯树着色的，且含 2^{k-1} 种不同的 4-着色。现在来考察 $k+1$ 的情况。设 $W_4 = v_2 - v_1v_4v_5v_3$ 是 J^{4k+1} 中的一个 4-轮，v_2 是轮心，如图 6(b)所示。对 W_4 中的哑铃 $X_4 = \Delta v_1v_2v_3 \cup \Delta v_2v_4v_5$ 施行哑铃变换(见图 6(a)与 6(c))。假设变换后的所得之图 $\zeta_{334}^{+}(J^{4k+1})$ 是可圈着色的，并设 $f \in C_4^0(\zeta_{334}^{+}(J^{4k+1}))$ 含 2-色圈，于是，存在两种情况：一种是 $f(v_2') = f(v_2)$，如图 6(a)所示；另一种是 $f(x_1) = f(x_2)$，如图 6(c)所示。对这两种情况实施哑铃收缩变换均得到图 J^{4k+1}。**注意到**：哑铃收缩变换是先实施一次缩 4-轮运算，然后再实施两次缩 3-轮运算，所得之图的自然着色(即每个顶点的着色是在 f 下的限制) 含 2-色圈，这与 J^{4k+1} 是纯树着色极大平面图矛盾！从而证明了 $\zeta_{334}^{+}(J^{4k+1})$ 是纯树着色的。

从图 6 可看出，对 J^{4k+1} 中的每个 4-轮，在轮圈上的着色被确定时，不失一般性，假设其顶点着色如图 6(b)所示。对该 4-轮实施哑铃变换后所得之图 $\zeta_{334}^{+}(J^{4k+1})$ 中，其哑铃收缩变换对象子图，在**外圈着**



(a) 9-阶 (b) 13-阶 (c) 17-阶 (d) 17-阶

图 4 阶数最小的 4 个哑铃极大平面图

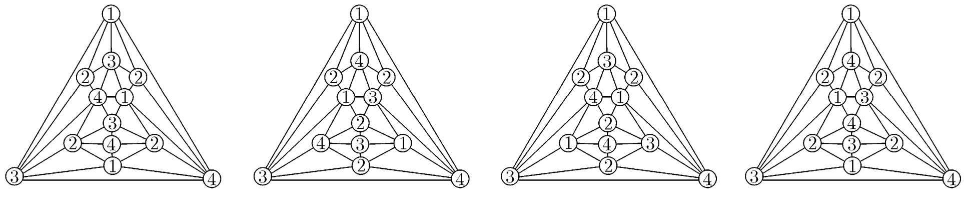


图 5 13-阶哑铃极大平面图的所有 4 种着色

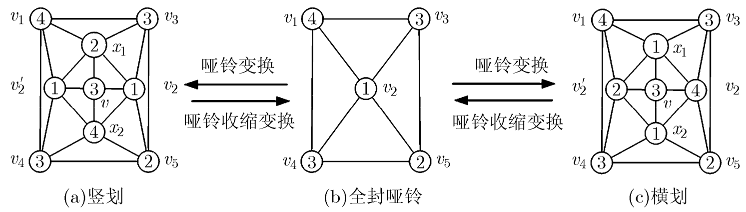


图 6 顶点赋色的哑铃变换与哑铃收缩变换

色不变的情况下，恰有两种着色，如图 6(a)与 6(c)所示。故 $\zeta_{334}^{+}(J^{4k+1})$ 中恰含有 $2^{k-1}\times 2=2^k$ 种不同的着色。从而本定理获证。

3.2.4 哑铃极大平面图的计数　9-阶哑铃极大平面图仅有 1 个，由于它的 3 个 4-轮是等同的，因而，13-阶的哑铃极大平面图也只有 1 个；在对 9-阶极大平面图实施哑铃变换时，由于只对其中一个实施哑铃变换，故其余两个是等同的，从而导致在 13-阶哑铃极大平面图中，它的 3 个 4-轮中有两个是等同的，由此推出 17-阶哑铃极大平面图共有 2 个，分别如图 4(c), 4(d)所示。进而，关于一般阶数的哑铃极大平面图，我们有

定理 2　记 t_k 为所有 $(4k+9)$-阶哑铃极大平面图的数目，$k\ge 0$，则有

$$t_k=\frac{(k+3)^2}{12}-\frac{7}{72}+\frac{(-1)^k}{8}+\frac{2}{9}\cos\frac{2k\pi}{3}$$

证明见文献[41]。

4　递归极大平面图

除四色猜想外，唯一 4-色极大平面图猜想已有 43 年的历史，业已成为图着色理论中一个很有影响的猜想。而此猜想的对象是递归极大平面图，故在本节对此类图进行深入地研究。

所谓递归极大平面图，是指从 K_4 出发，不断地在三角面上嵌入 3 度顶点得到的极大平面图。这里所言在一个三角形面上**嵌入 3 度顶点**是指：首先在该面上添加一个顶点，然后让该顶点与这个三角形面上的每个顶点相连边。用 Λ 表示所有递归极大平面图构成的集合，而全体 n-阶递归极大平面图构成的集合记为 Λ_n，并令 $\gamma_n=|\Lambda_n|$。易证，当 $n=4,5,6$ 时，γ_n=1，对应的递归极大平面图分别如图 7 所示。

4.1 基本性质

定理 3　设 G 是一个 n-阶递归极大平面图，则 G 至少含有 2 个 3-度顶点，并且当 $n\ge 5$ 时，任意两个 3-度顶点之间均不相邻。

证明　对顶点数 n 施行数学归纳法。当 $n=4, 5, 6$ 时，γ_4=γ_5=γ_6=1，对应于 Λ_4, Λ_5 与 Λ_6 中的递归极大平面图见图 7，结论显然成立。

假设当 $n\ge 6$ 时结论成立，即对于任意的 n-阶递归极大平面图 G，它至少含有两个 3-度顶点，且所有的 3-度顶点互不相邻。我们来考虑顶点数为 $n+1$ 的情况。

对于任意 $G\in\Lambda_{n+1}$，由于 G 是从 n 个顶点的递归极大平面图中某一个三角形构成的面上通过增加一个 3-度顶点 v 形成的，不妨设 $N_G(v)=\{v_1,v_2,v_3\}$。由归纳假设 $G-v$ 中至少含有两个 3-度顶点，且任意两个 3-度顶点都是不相邻的。对于图 $G-v$，若 $\{v_1,v_2,v_3\}$ 中含有 3-度顶点，那么最多含有一个 3-度顶点，不妨令为 v_1，显然除 v_1 外还至少存在一个 3-度顶点，且任意两个 3-度顶点互不相邻。由于 v 是 G 的一个 3-度顶点，且 v 与除 $\{v_1,v_2,v_3\}$ 外的点都不相邻，因此 G 中也至少含有 2 个 3-度顶点，且任意 2 个 3-度顶点也互不相邻，故此时结论成立；若 $\{v_1,v_2,v_3\}$ 中不含有 3-度顶点，结论依然成立。证毕

定理 4　(1)不存在恰有 2 个相邻的 3-度顶点的极大平面图；(2)不存在恰有 3 个两两相邻的 3-度顶点的极大平面图。

证明　(1)用反证法。设 G 是一个极大平面图，$u,v\in V(G)$，$d(u)=d(v)=3$ 且 $uv\in E(G)$。由于顶点 u 是 3-度顶点，故可设 $N(u)=\{v,x,y\}$。注意到 G 是极大平面图，因此顶点 u 所在的面是由顶点 v, x, y

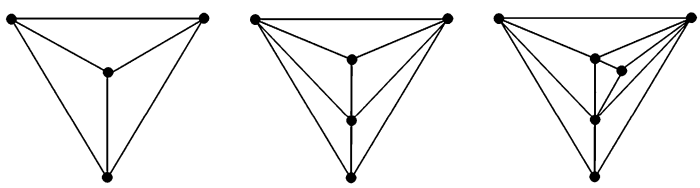


图 7 顶点数分别为 4, 5, 6 的 3 个递归极大平面图

构成的三角形，即顶点 v 与顶点 x,y 均相邻。于是形成一个子图 K_4，如图 8 所示。由于 G 是极大平面图，因此当 G 还有其它顶点时，顶点 u 或 v 必与其它顶点形成三角形，这与 $d(u)=d(v)=3$ 矛盾！若无其它顶点，K_4 显然是一个含有 4 个相邻的 3-度顶点的图。因此，不存在恰有 2 个相邻的 3-度顶点的极大平面图。类似可证(2)。 证毕

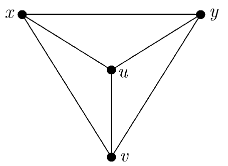


图 8 定理 4 证明的示意图

定理 5 设 G 是一个极大平面图，且 G 只有一个 3-度顶点，则通过逐点删除 3-度顶点的方法，可得到一个不含 3-度顶点的子图。

证明 令 v 是图 G 中唯一的 3-度顶点，其邻域 $N_G(v)=\{u_1,u_2,u_3\}$。则 u_1, u_2, u_3 构成了一个三角形。令 $G_1=G-v$，则 G_1 也是一个极大平面图，有 4 种可能的情况：

(1) $\delta(G_1)\geq 4$；

(2)只有 1 个 3-度顶点；

(3)只有 2 个 3-度顶点；

(4)只有 3 个 3-度顶点。

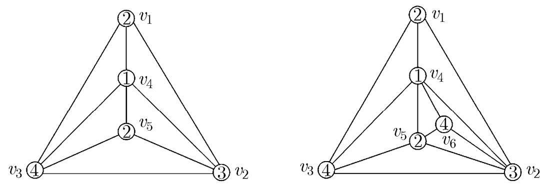
对于第 1 种情况显然结论成立；而由定理 4 知第 3, 4 种情况不存在。故只考虑第 2 种情况。即在子图 G_1 中只存在一个 3-度顶点，记作 u_1。令 $G_2=G_1-u_1$，类似于上述分析方法，若 $\delta(G_2)\geq 4$，则结论得证；否则，G_2 必恰有一个 3-度顶点。如此下去，在有限步内，必有 $\delta(G_m)\geq 4$；否则，当 G_m 只含有 4 个顶点时只能同构于 K_4，从而说明图 G 是递归极大平面图，但 G 只有一个 3-度顶点，与定理 3 矛盾！故本定理获证。

定理 6 设 G 是一个阶数 ≥ 5 的递归极大平面图，v 是其中的一个 3-度顶点。则 $G-v$ 仍是递归极大平面图。

证明 设 G 是从 $K_3=v_1v_2v_3$ 开始，依次添加 3-度顶点 $v_4,v_5,\cdots,v_n$ 得到的递归极大平面图。若 $v\in\{v_4,v_5,\cdots,v_n\}$，当 $v=v_n$ 时，显然 $G-v$ 是递归极大平面图；当 $v=v_i$，$4\leq i\leq n-1$ 时，则在 $G-v$ 中，依次删去 3-度顶点 $v_n,v_{n-1},\cdots,v_{i+1}$，所得到的图记为 G'。易验证，G' 可以从 G 开始，依次删去 3-度顶点 $v_n,v_{n-1},\cdots,v_{i+1},v_i$ 得到，故 G' 是递归极大平面图。所以，$G-v$ 仍是递归极大平面图。若 $v\in\{v_1,v_2,v_3\}$，不妨设 $v=v_1$，则 $G-v$ 是在三角形 $\Delta v_2v_3v_4$ 的基础上依次添加 3-度顶点 $v_5,v_6,\cdots,v_n$ 得到的递归极大平面图。 证毕

4.2 (2,2)-递归极大平面图

本小节引入一类特殊的递归极大平面图：(2,2)-递归极大平面图，并研究它的相关性质。一个递归极大平面图 G 称为**(2,2)-递归极大平面图**，如果 G 中只有 2 个度数为 3 的顶点，且这两个顶点之间的距离为 2。容易证明 5-阶(2,2)-递归极大平面图只有 1 个，如图 9(a)所示，6-阶(2,2)-递归极大平面图也只有 1 个，如图 9(b)所示。



(a) 5-阶(2,2)-递归极大平面图 (b) 6-阶(2,2)-递归极大平面图

图 9 5-阶及 6-阶(2,2)-递归极大平面图

为了弄清(2,2)-递归极大平面图的结构，我们先将 4-阶完全图 K_4 分成 3 个区，并给出相应顶点的名称，由顶点 v_1,v_2,v_3 标定的三角形称为**外三角形**，顶点 u 称为**中心顶点**或简称为**中心点**，如图 10 所示。我们约定：顶点 v_1 着色为颜色 2，顶点 v_2 着颜色 3，顶点 v_3 着颜色 4，**顶点** $u(v_4)$ 着颜色 1，称这 4 个顶点与对应的着色为(2,2)-递归极大平面图的**色坐标系**中的**基本坐标轴**。4 个色坐标轴分别为 u (颜色 1)，v_1 (颜色 2)，v_2 (颜色 3)，v_3 (颜色 4)。

显然，没有 4-阶(2,2)-递归极大平面图；不同构的 5-阶(2,2)-递归极大平面图只有 1 个，就是在如图 10 所示 K_4 的Ⅰ区、Ⅱ区或Ⅲ区中通过嵌入一个 3-度顶点的运算(即扩 3-轮运算)而得到。不失一般性，我们约定，所增加的顶点在Ⅱ区。因而该顶点着色为颜色 2(见图 9(a))；6-阶不同构的(2,2)-递归极大平面图也只有 1 个(如图 9(b)所示)，因为在 5-阶极大平面图的任意面内嵌入一个 3-度顶点所得到的 6-阶极大平面图均是同构的。故这里约定：在由顶点 v_2,v_4,v_5 这 3 个顶点构成的面上(即在Ⅱ区的子Ⅰ区)嵌入第 6 个顶点，显然，它着色为颜色 4，如图 9(b)

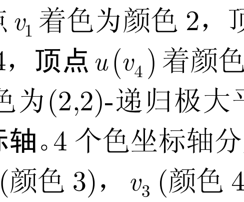


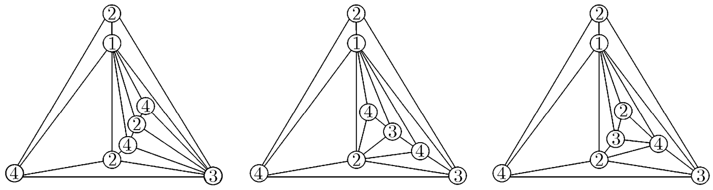
图 10 色坐标系的基本框架图

所示。不失一般性，在此进一步约定更高阶数的(2,2)-递归极大平面图只在Ⅰ区和Ⅱ区内有顶点，在Ⅲ区内无顶点。

在上述约定的基础上，现在来讨论(2,2)-递归极大平面图的分类。有两种分类方法：

第 1 种方法是按照嵌入 3-度顶点的区来分类：(1)只在Ⅱ区通过不断嵌入 3-度顶点而得到的(2,2)-递归极大平面图，如图 11 所示的 3 个图均是此种类型；(2)通过不断地在Ⅰ区和Ⅱ区之间随机地嵌入 3-度顶点而得到，如图 12 所示。对于极大平面图，有如下结论。

命题 1　极大平面图的任一面均可成为无穷面。



(a)相邻型　(b)非相邻型　(c)非相邻型

图 11 只在Ⅱ区嵌入 3-度顶点的(2,2)-递归极大平面图

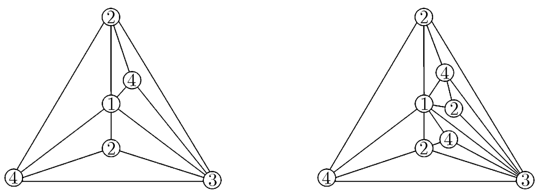


图 12 在Ⅰ区与Ⅱ区之间随机嵌入 3-度顶点的(2,2)-递归极大平面图

由命题 1 可知，在上述Ⅰ区与Ⅱ区之间随机嵌入 3-度顶点的(2,2)-递归极大平面图中，可以将Ⅰ区或者Ⅱ区中的任一 3-度顶点变换到外三角形面上，就等价于分类中的(1)的情况，即只在Ⅱ区通过不断地嵌入 3-度顶点而得到的(2,2)-递归极大平面图。因此，只需考虑在Ⅱ区通过不断嵌入 3-度顶点而得到的(2,2)-递归极大平面图即可。

第 2 种分类方法是根据 2 个 3-度顶点所在长度为 2 路中的 3 个顶点是否存在一个公共的相邻顶点来进行分类：若存在，则称为**相邻型**，否则称为**非相邻型**。如图 11(a)是相邻型的，而图 11(b)，11(c)均是非相邻型的。

由图 9(a)可以看出，5-阶(2,2)-递归极大平面图是一个**双心轮图**，且每个轮心的邻域中顶点度数均为 4，但当阶数≥ 6时，有下述结论。其中**双心轮图**是指一个圈与两个孤立点的联图构成的极大平面图。

定理 7　(1)设G是一个阶数≥ 6的(2,2)-递归极大平面图，则对G中每个 3-度顶点v，其邻域$N(v)$中恰有一个顶点的度数为 4；(2)每个非相邻型n(≥ 7)-阶(2,2)-递归极大平面图G有且只有一个度数为$n-1$的顶点，称此顶点为图G的**中心顶点**，记作u。并且在(2,2)-递归极大平面图的任意 4-色组划分中，有且只有中心顶点u着颜色 1；(3)只在Ⅱ区嵌入 3-度顶点的相邻型(2,2)-递归极大平面图的任意 4-色组划分中，不仅只有中心顶点u着颜色 1，而且有且只有色坐标轴的顶点v_2着颜色 3。

证明　(1)采用数学归纳法。由于阶数为 5 的极大平面图只有一个(如图 9(a)所示)，它是一个双心轮图，故每个三角形面是等同的。因此，同构意义下的 6-阶递归极大平面图只有一个，且该递归极大平面图是(2,2)-递归极大平面图(如图 9(b)所示)。该图中的两个 3-度顶点的邻域中均恰有一个顶点的度数等于 4，其余顶点的度数均≥ 5，这就证明了当顶点数$n=6$时结论成立。

假设当阶数$n\geq 6$时结论成立，我们来考察顶点数为$n+1$的情况。设G是一个阶数为$n+1$的(2,2)-递归极大平面图，且v是G中的一个 3-度顶点。分两种情况讨论：

情况 1　$N(v)$中含有 2 个或 3 个 4-度顶点；则$G-v$也是一个阶数$n\geq 6$的递归极大平面图，它含有 2 个或者 3 个两两相邻的 3-度顶点，这与定理 3 矛盾；

情况 2　$N(v)$中不含有 4 度顶点，则$G-v$也是一个阶数$n\geq 6$的递归极大平面图，且只有一个 3-度顶点，这与定理 3 矛盾。

综上两种情况，我们证明了 3-度顶点v的邻域$N(v)$中有且仅有一个 4 度顶点。

(2)和(3)可通过在逐步构造(2,2)-递归极大平面图的过程中获证。　证毕

推论 1　设G是一个阶数≥ 6的(2,2)-递归极大平面图，任取G的一个 3-度顶点v，则$G-v$仍然是(2,2)-递归极大平面图。

定理 8　设η_n表示n-阶非同构(2,2)-递归极大

平面图的数目，$n \geq 6$，则$\eta_n \leq 2^{n-6}$。特别，$\eta_6 = 1$，$\eta_7 = 2$，$\eta_8 = 3$，$\eta_9 = 6$。

证明 由于任意$(2,2)$-递归极大平面图可通过在6-阶的$(2,2)$-递归极大平面图的II-区添加3-度顶点得到，当在i-阶的$(2,2)$-递归极大平面图中添加第$i+1$个顶点时，$6 \leq i \leq n-1$，对于每个非相邻型的，只有两个三角形面可选择；对于相邻型的，有3个三角形面可选择，但易验证在与4-度顶点关联的两个面中添加顶点所得到的图是同构的。因此，$\eta_n \leq 2^{n-6}$。当$n \leq 9$时，其相应的7-阶，8-阶及9阶图分别如图13所示。 证毕

4.3 扩4-轮运算图的着色

不失一般性，今后总是假定(2,2)-递归极大平面图G就是只在Ⅱ区嵌入3-度顶点得到的(2,2)-递归极大平面图。因此，一个(2,2)-递归极大平面图可唯一地由它的颜色序列来表示。具体表示方法如下：

设$V(G) = \{v_1, v_2, v_3, v_4 = u, v_5, \cdots, v_n\}$，顶点$v_1 = x$表示第1个固定的3-度顶点，顶点$v_n = y$表示第2个3-度顶点；顶点$v_1 = x$，$v_2$，$v_3$和$v_4 = u$分别表示第1，第2，第3和第4个色坐标轴，顶点$v_4 = u$是中心顶点；顶点$v_{n-1}$表示$G_{n-1} = G - v_n$中的3-度顶点；顶点$v_{n-2}$表示子图$G_{n-2} = G_{n-1} - v_{n-1}$的3-度顶点；依此类推。我们用序列$c_1c_2\cdots c_n$来表示顶点$v_1, v_2, v_3, v_4 = u, v_5, \cdots, v_n$的颜色序列，其中$c_i$表示顶点$v_i$在$(2,2)$-递归极大平面图$G$中所着的颜色：$c_i \in \{1,2,3,4\}$。根据(2,2)-递归极大平面图$G$的定义，易知此表示方法也唯一地确定了一个图的结构，即从如图10所示的K_4出发，按照每个顶点所着的颜色来选择该顶点嵌入的三角面。

例如，对于色序列为$c_1c_2c_3c_4c_5c_6c_7c_8c_9 = 234124323$，容易分析它所对应的(2,2)-递归极大平面图是图13中所示的第3行第2个图。

关于(2,2)-递归极大平面图的色序列，容易得到下述结论：

定理 9 设$c_1c_2\cdots c_n$表示(2,2)-递归极大平面图G的色序列。在“只在Ⅱ区嵌入3-度顶点”约定下，该序列的前6个顶点颜色是确定的，即为$c_1 = 2$，$c_2 = 3$，$c_3 = 4$，$c_4 = 1$，$c_5 = 2$，$c_6 = 4$；当G为相邻型时，$c_7 = 2$；而当G为非相邻型时，$c_7 = 3$或2。

证明 基于图10知：$c_1 = 2$，$c_2 = 3$，$c_3 = 4$，$c_4 = 1$；基于图9(a)知：$c_5 = 2$；基于图9(b)知：$c_6 = 4$；进而由图9(b)，易证c_7的情况成立。 证毕

下面讨论扩4-轮运算在(2,2)-递归极大平面图类中顶点着色问题。我们知道，对一个给定的(2,2)-递归极大平面图G，它是唯一4-可着色的，且每个顶点所着的颜色也是确定的。

设图G是一个(2,2)-递归极大平面图，f是它的唯一4-着色，xuy是它的一条2-长路。显然，在对图G实施关于路xuy的扩4-轮运算后所得的图$\zeta_4^+(G)$中存在如下一种着色，记为f^*，其中v是扩4-轮得到的4-轮轮心，u'是该4-轮的轮圈上新添加的顶点，

$$f^*(z) = \begin{cases} f(u), & z = u' \text{ 或 } u \\ \{1,2,3,4\} \setminus \{f(x), f(u), f(y)\}, & z = v \\ f(z), & \text{其它} \end{cases}$$

即，f^*是图$\zeta_4^+(G)$的一种着色，它使得顶点u与u'着相同颜色，顶点v的着色与顶点x, u, y不同，其余顶点的着色与G中f下的着色相同。我们称f^*为图

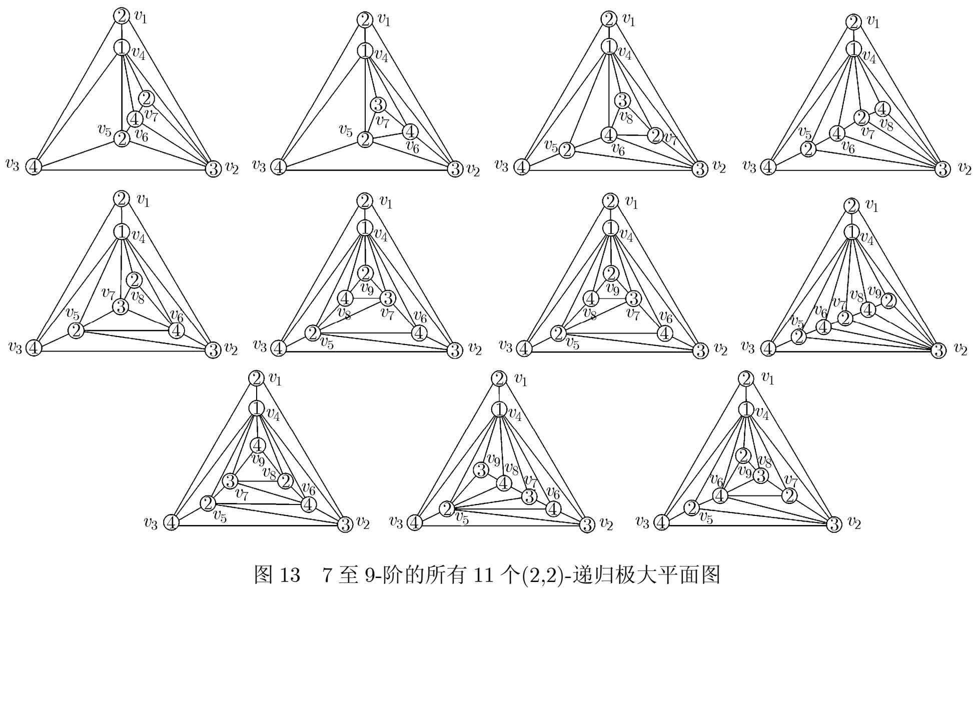


图13 7至9-阶的所有11个(2,2)-递归极大平面图

$\zeta_4^+(G)$ 的**自然 4-着色**。

自然要问：经过扩 4-轮运算后所得到的图 $\zeta_4^+(G)$ 是否仍是唯一 4-可着色的？事实上，答案是否定的，即 $|C_4^0(\zeta_4^+(G))| \ge 2$。

定理 10　设 G 是一个 n-阶(2,2)-递归极大平面图。x 与 y 是它的两个 3-度顶点，且 u 是中心点，则对 G 实施基于路 xuy 的扩 4-轮运算后得到的图 $\zeta_4^+(G)$ 不是唯一 4-可着色的。

证明　当 $f(x)=f(y)$ 时，显然成立。下面考虑 $f(x)\ne f(y)$ 的情况，设 $N(x)=\{u,x_1,x_2\}$，$f(u)=1$，$f(x)=2$，$f(x_1)=3$，$f(x_2)=4$。由定理 7 知，G 中只有顶点 u 着颜色 1，故 y 只能着颜色 3 或 4。不失一般性，假设 $f(y)=f(x_1)=3$。由于 G 是(2,2)-递归极大平面图，即 G 可通过在 $K_4=G[\{x,u,x_1,x_2\}]$ 中依次添加 $n-4$ 个 3-度顶点 $v_1,v_2,\cdots,v_{n-4}$ 得到。令 $v_1,v_2,\cdots,v_{n-4}$ 中与 x_1 相邻的下标最大顶点为 w，则 $f(w)=2$ 或 4。

在图 $\zeta_4^+(G)$ 中，设新添加顶点分别为 v (扩 4-轮得到的 4-轮轮心)和 u'，$N(v)=\{x,y,u,u'\}$，其中 u 与 x_1 相邻，u' 与 x_2 相邻。由自然着色定义知，$f^*(w)=f(w)=2$ 或 4。

若 $f^*(w)=2$，则删去顶点 x,x_1,v 上的颜色，将 u 所在的 14-分支(即，所有着颜色 1 和 4 的顶点导出子图的某个分支)上的颜色互换，再给 x,x_1,v 分别着颜色 3,1,2，易证，所得到的着色是 $\zeta_4^+(G)$ 的一种异于 f^* 的 4-着色。若 $f^*(w)=4$，则删去顶点 x,x_1 上的颜色，将 u 所在的 12-分支上的颜色互换，再给 x,x_1 分别着颜色 3,1，易验证，所得到的着色是 $\zeta_4^+(G)$ 的一种异于 f^* 的 4-着色。　证毕

注：定理10说明了对(2,2)-递归极大平面图 G 施行基于 2-长路 xuy 的扩 4-轮运算后得到的图 $\zeta_4^+(G)$ 不是唯一 4-可着色的，其中 x,y 必须都是 G 中的 3-度顶点。当 x,y 不是 3-度顶点时，$\zeta_4^+(G)$ 有可能还是唯一 4-可着色的。

5　唯一 4-色极大平面图的证明思路

唯一 4-色极大平面图猜想是一个尚待解决的难题，该猜想的对象是递归极大平面图，故在第 4 节中对此类图的性质展开了详细讨论，我们在文献[41]中提出了纯树着色猜想，并指出若此猜想成立，则唯一 4-色极大平面图猜想成立。特别在本文第 3 节中重点针对哑铃极大平面图进行了深入研究。本节给出的唯一 4-色极大平面图猜想证明思路实际上是证明纯树着色极大平面图猜想。

设 G 是一个纯树着色平面图，W_4，W_5 分别是 G 中的一个 4-轮和 5-轮。用 τ_i，$i=1,2,3$，及 τ_1' 表示多米诺构形的 4 个生成运算算子[39]，用 $\tau_i(W_4)$ 表示对 G 中的 4-轮 W_4 实施 τ_i 运算后所得之图，其余记号类似，这里不再一一介绍。

下面，给出纯树着色猜想的证明思路，即按照如下 9 种情况，逐一给出证明：

(1)若 G 是纯树着色极大平面图，则 $\tau_1(W_4)$ 是可圈着色的；

(2)若 G 是纯树着色极大平面图，则 $\tau_1(W_5)$，$\tau_2(W_4)$ 是可圈着色的；

(3)若 G 是纯树着色极大平面图，则 $\tau_2(W_5)$ 是可圈着色的；

(4)若 G 是纯树着色极大平面图，则 $\tau_3(W_5)$ 是可圈着色的；

(5)若 G 是纯树着色极大平面图，则 $\tau_1(\tau_1(W_4))$ 是可圈着色的；

(6)若 G 是纯树着色极大平面图，则 $\tau_2(\tau_1(W_4))$ 及 $\tau_1(\tau_2(W_4))$ 是可圈着色的；

(7)若 G 是纯树着色极大平面图，则 $\tau_1'(\tau_2(W_4))$ 是可圈着色的；

(8)若 G 是纯树着色极大平面图，则 $\tau_1'(\tau_2(W_5))$ 是可圈着色的；

(9)若 G 是纯树着色极大平面图，则 $\tau_2(\tau_2(W_4))$ **是纯树着色的**。

有兴趣的读者可参见文献[39]中的 3.4 节，特别是图 17。关于这方面的详细论述将在本系列后续文章中给出。

6　结束语

本文主要对图着色理论中另一个至今未被解决的猜想——“唯一 4-色极大平面图猜想”展开研究。由于此猜想的对象是递归极大平面图，所以，我们在本文的第 4 节里对此类图进行了深入地研究。

我们在文献[41]中所提出的纯树着色猜想是：“一个极大平面图 G 是纯树着色的充分必要条件是 G 是正二十面体或哑铃极大平面图”，并指出：若纯树着色猜想成立，则唯一 4-色极大平面图猜想成立。故本文的另一个主要内容是研究哑铃极大平面图结构与性质。

本文所提出的唯一 4-色极大平面图猜想的证明思路实际上是给出证明纯树着色猜想的思路。该证明思路是：在假设 G 是纯树着色极大平面图的基础上，基于文献[39]中所给出的**扩缩运算法**，以及任意 $n(\ge 9)$-阶最小度 ≥ 4 的极大平面图要么有父代图，要么有祖父图，我们给出了证明纯树着色猜想的 9 种情况，其中 8 种情况是否定的，只有一种情况是肯定的。

本文的工作为证明纯树着色猜想奠定了一定的基础。在本系列后续文章中，我们将给出 4-色极大平面图的扩缩运算系统，简称为**色扩缩运算系统**。然后在此基础上，纯树着色猜想有望得到完整的证明。

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Theory on Structure and Coloring of Maximal Planar Graphs

(3)Purely Tree-colorable and Uniquely 4-colorable Maximal Planar Graph Conjectures

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Abstract: A maximal planar graph is called the recursive maximal planar graph if it can be obtained from K_4 by embedding a 3-degree vertex in some triangular face continuously. The uniquely 4-colorable maximal planar graph conjecture states that a planar graph is uniquely 4-colorable if and only if it is a recursive maximal planar graph. This conjecture, which has 43 years of history, is a very influential conjecture in graph coloring theory after the Four-Color Conjecture. In this paper, the structures and properties of dumbbell maximal planar graphs and recursive maximal planar graphs are studied, and an idea of proving the uniquely 4-colorable maximal planar graph conjecture is proposed based on the extending-contracting operation proposed in this series of article (2).

Key words: Uniquely 4-colorable maximal planar graph conjecture; Purely tree-colorable planar graph conjecture; Dumbbell maximal planar graphs; Recursive maximal planar graphs

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1 Introduction

Graph theory originated from the KÖNIGSBERG seven bridges problem studied by EULERIN 1736, and the electric network problem studied by KIRCHHOFF in 1847. In the last 70 years, graph theory was rapidly developed. To its reason, one is influenced by the development of the electronic computer; the other more important is the Four-Color Conjecture. Specifically, due to the Four-Color Conjecture, many areas of graph theory were created, such as topological graph theory, maximum independent set and maximum clique theory, vertex and edge covering theory, chromatic polynomial theory, Tutte-polynomial theory, factor theory and integer flow theory, especially the graph coloring theory and so on.

In the field of graph coloring, apart from the famous Four-Color Conjecture, researchers have proposed many conjectures, including the uniquely 4-colorable maximal planar graph conjecture discussed in this paper. This conjecture was proposed by GREENWELL and KRONK[1] in 1973, which is closely related to the Four-Color Conjecture, and has not been resolved yet.

The concept of uniquely colorable graphs was proposed by GLEASON and CARTWRIGHT[2], and CARTWRIGHT and HARARY[3] successively. CARTWRIGHT and HARARY obtained some sufficient conditions for determining whether a labeled graph is uniquely colorable or not. Since then, many researches have been done in this field. For example, HARARY, *et al*[4] studied the connectivity and the number of edges of uniquely k-colorable graphs. Furthermore, many scholars studied on the problem that whether there exists a uniquely k-colorable graph without K_3 for $k \geq 3$. NEŠETIL[5,6] studied the properties of critical-uniquely colorable graphs and proved that there exist uniquely k-colorable graphs without triangles. In 1974, GREENWELL and LOVÁSZ[7] proved that there exist uniquely k-colorable graphs without short odd cycles; and in 1975, MÜLLER[8] solved the general case of this problem by the method of construction (also see Ref. [9]), that is, for any integer $k \geq 3$ and t, there exists a uniquely k-colorable graph with girth greater than t. MÜLLER[8,9], AKSIONOV[10], MELNIKOV and STEINBERG[11] investigated the edge-critical

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uniquely colorable graphs; WANG and ARTZY[12] showed that for $k \geq 3$, if there exists a uniquely k-colorable graph without K_3, then the number of edges of the graph is greater than $k^2 + k - 1$; OSTERWEIL[13] constructed a kind of uniquely 3-colorable graphs by the so-called 6-cliquerings. Moreover, he demonstrated how to use this technique to construct uniquely k-colorable graphs $(k > 3)$; BOLLOBÁS and SAUER[14] proved that for all $k \geq 2$ and $g \geq 3$, there exists a uniquely k-colorable graph with girth at least g. Also, they proved that for any $k \geq 3$ and n, there always exists a critical-uniquely k-colorable graph with at least n vertices; DMITRIEV[15] generalized the result of BOLLOBÁS[16]; XU[17] proved that if G is a uniquely k-colorable graph with order n and size m, then $m \geq (k-1)n - (1/2)k(k-1)$, and showed this bound is the best possible. Furthermore, he conjecture that if G is a uniquely k-colorable graph with order n and size $(k-1)n - (1/2)k(k-1)$, then G contains K_k. At the same time, CHAO and CHEN[18] showed that for any integer $n \geq 12$, there exists a uniquely 3-colorable graph of order n without triangles; AKBARI, *et al*[19] proved that there exists a K_3-free uniquely 3-colorable graph G with 24 vertices and $\mathrm{SH}(G) = 45$ edges, where $\mathrm{SH}(G) = (k-1)n - (1/2)k(k-1)$. This result disproved XU's conjecture[17].

In terms of the uniquely edge-colorable graphs, GREENWELL and KRONK[1] studied this problem first in 1973. They proposed a conjecture as follows.

Conjecture 1 If G is a uniquely 3-edge-colorable cubic graph, then G is a planar graph that contains a triangle.

In 1975, FIORINI[20] independently studied the problem of uniquely edge-colorable graphs, and obtained some similar results as GREENWELL and KRONK[1]. After that, many scholars involved in this field, such as THOMASON[21,22], FIORINI and WILSON[23], ZHANG[24], GOLDWASSER and ZHANG[25,26], and KRIESSELL[27].

In 1977, FIORINI and WILSON[23], and FISK[28] independently proposed the following conjecture, respectively.

Conjecture 2 Any uniquely 3-edge-colorable cubic planar graph with at least 4 vertices contains a triangle.

This conjecture was proposed based on Conjecture 1. FOWLER[29] also investigated this conjecture in detail. However, it is still open.

As for uniquely colorable planar graphs, CHARTRAND and GELLER[30] completed many important works in 1969. They proved that any uniquely 3-colorable planar graph of order $n \geq 4$ contains at least two triangles, any uniquely 4-colorable planar graph is a maximal planar graph, and there is no uniquely 5-colorable planar graph.

What is the necessary and sufficient condition for a planar graph to be uniquely 3-colorable? This problem is still open. Many scholars studied the properties of uniquely 3-colorable planar graphs. In 1977, AKSIONOV[31] proved that uniquely 3-colorable planar graphs with order ≥ 6 contain at least 3 triangles, also described the structure of those graphs contain exactly 3 triangles. In the same year, MELNIKOV and STEINBERG[11] investigated the edge-critical uniquely 3-colorable planar graphs and proposed the following problem: find the exact upper bound $\mathrm{size}(n)$ of edges in edge-critical uniquely 3-colorable planar graphs with order n. In 2013, MATSUMOTO[32] proved $\mathrm{size}(n) \leq 8/3n - 17/3$. Recently, LI, *et al*[33,34] proved $\mathrm{size}(n) \leq 5/2n - 6,\ n \geq 6$, and showed that there exist adjacent triangles in uniquely 3-colorable planar graphs containing at most 4 triangles.

Naturally, a problem will be raised that which maximal planar graphs are uniquely 4-colorable? In other words, what is the characteristic of uniquely 4-colorable planar graphs? Obviously, this problem is the key to study uniquely 4-colorable planar graphs. So far, many scholars have been investigating this problem[24,35–37].

Indeed, FISK[28] proposed a conjecture equivalent to Conjecture 2.

Conjecture 3 A planar graph G is uniquely 4-colorable if and only if G is a recursive maximal planar graph.

It can be checked that Conjecture 2 and Conjecture 3 are equivalent, and both of them are the special case of Conjecture 1. In 1998, BOHME, *et al* [35] proved that the minimum counterexample of this conjecture is 5-connected. We call

Conjecture 3 the uniquely 4-colorable maximal planar graph conjecture.

All graphs considered in this paper are finite, simple, and undirected. For a given graph G and a vertex v of G, $V(G)$, $E(G)$, $d_G(v)$, and $N_G(v)$ denote the vertex set, the edge set, the degree of v, and the neighborhood of v (the set of all vertices adjacent to v) respectively, which are written as V, E, $d(v)$, and $N(v)$ for short. The cardinality $|V(G)|$ of the set $V(G)$ is called the order of G. For a graph $H=(V',E')$, if $V'\subseteq V$, $E'\subseteq E$, then H is called a subgraph of G. Whenever $u,v\in V(H)$ are adjacent in the graph G, they are also adjacent in the graph H, then H is called an induced subgraph of G. An induced subgraph of G with vertex set V' is denoted by $G[V']$. By starting with a disjoint union of G and H and joining every vertex of G to every vertex of H by adding edges, one obtains the join of G and H, denoted by $G\vee H$. We use K_n to denote the complete graph of order n. The join $C_n\vee K_1$ of a cycle and a single vertex is referred to as a wheel with n spokes, denoted by W_n, where C_n and K_1 are called the cycle and the center of the wheel, respectively. If $V(K_1)=\{x\}$, we also write the cycle C_n of the wheel W_n as C^x.

A k-vertex coloring, or simply a k-coloring, of a graph G is a mapping f from V to the color set $C(k)=\{1,2,\cdots,k\}$ such that $f(x)\neq f(y)$ if $xy\in E(G)$. A graph G is k-colorable if it has a k-coloring. The minimum k for which a graph G is k-colorable is called its chromatic number, denoted by $\chi(G)$. If $\chi(G)=k$, then G is called a k-chromatic graph. Alternatively, each k-coloring f of G can be viewed as a partition $\{V_1,V_2,\cdots,V_k\}$ of V, say a k-color class partition of G, where V_i denotes the set of vertices assigned the color i, *i.e.*

$$V(G)=\bigcup_{i=1}^{k}V_i,\ V_i\neq\phi,\ V_i\cap V_j=\phi,\ i\neq j,\ i,j=1,2,\cdots,k$$

Obviously, V_i is an independent set of G, $i=1,2,\cdots,k$; also we call V_i a color class of f. The set of all k-colorings of a graph G is denoted by $C_k(G)$. For a k-chromatic graph G, we use $C_k^0(G)$ to denote the set of all k-color class partitions of G, which is called k-color class partition set of G.

A k-chromatic graph G is called uniquely k-colorable if all k-colorings of G induce the same partition of the vertex set $V(G)$ into the k independent sets.

Similarly, the concept of edge coloring of a graph can be given, that is, an assignment from the color set to its edge set such that no two adjacent edges have the same color. A graph G is uniquely k-edge colorable if there is exactly one partition of the edge set $E(G)$ into k matchings.

A maximal planar graph is a planar graph to which no new edges can be added without violating the planarity. A triangulation is a planar graph in which every face is bounded by three edges (including its infinite face). It can be proved that a maximal planar graph is equivalent to a triangulation.

Let G be a maximal planar graph, $\mathrm{Aut}(G)$ denote the automorphism group of G, and H, H' be two isomorphic subgraphs of G. If there exists a $\sigma\in\mathrm{Aut}(G)$ such that $\sigma(H)=H'$, then H and H' are said to be identical; otherwise, H and H' are unidentical.

The concepts and symbols not given in this paper can be viewed in Refs. [38,39].

2 Tree-colorings and Cycle-colorings

Let G be a 4-colorable maximal planar graph, $C(4)=\{1,2,3,4\}$ be the color set, and $f\in C_4^0(G)$. If there exists a cycle $C_{2m}=v_1v_2\cdots v_{2m}$ in G such that $|\{f(v_1),f(v_2),\cdots,f(v_{2m})\}|=2$, then we refer to C_{2m} as a bicolored cycle of f, or say f contains a bicolored cycle, and we call f a cycle-coloring of G. If the colors on C_{2m} are i and t, then we call C_{2m} an it-cycle. On the other hand, if $f\in C_4^0(G)$ contains no bicolored cycle, then f is called a tree-coloring of G. For the 4-colorings of the graph shown in Fig. 1, f_1, f_2 are cycle-colorings and f_3, f_4 are tree-colorings. All maximal planar graphs can be divided into three categories according to the cycle-colorings and tree-colorings: pure tree-coloring graphs, namely such graphs have only tree-colorings; pure cycle-coloring graphs, namely these graphs have only cycle-colorings; impure coloring graphs that have both cycle-colorings and tree-colorings. For example, the graph shown in Fig.1 belongs to impure coloring graphs.

A maximal planar graph G is cycle-colorable if there exists a $f \in C_4^0(G)$ such that f is a cycle-coloring, and tree-colorable if there exists a $f \in C_4^0(G)$ such that f is a tree coloring. A statistics on the number of 4-colorings of non-separable maximal planar graphs with order $n(7 \le n \le 11)$ and minimum degree ≥ 4 shows that the proportion of tree-colorings is about 2%, that is to say the proportion of cycle-colorings is about 98%. At present, there are few studies on the structure and properties of the tree-colorable maximal planar graphs. In Ref. [40], ZHU, *et al* proved that every tree-colorable maximal planar graph G with minimum degree ≥ 4 contains at least 4 vertices of odd degree, and when G contains exactly 4 vertices of odd degree, the induced subgraph of the 4 vertices contains no triangles and is not the claw. There are totally 54 maximal planar graphs with order $n(7 \le n \le 11)$ and minimum degree ≥ 4, in which the graph shown in Fig.4(a) is the only purely tree-colorable graph, called the 9-dumbbell maximal planar graph and denoted by J^9. We also call this graph the basic dumbbell maximal planar graph. For more information, readers can refer to Ref. [41].

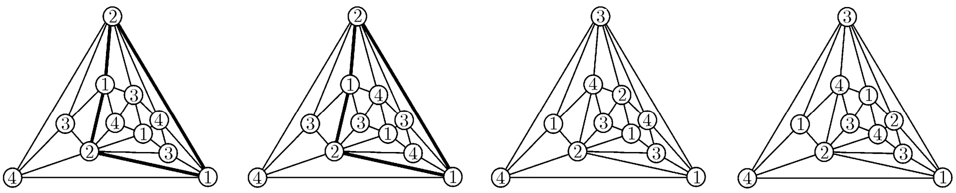
3 Purely Tree-colorable Maximal Planar Graphs

In Section 2, we partitioned maximal planar graphs into three classes: purely tree-colorable graphs, purely cycle-colorable graphs, and impure colorable graphs. It is difficult to give a necessary and sufficient condition to determine which class a maximal planar graph belongs to, even to describe the structure of these three kinds of maximal planar graphs. In fact, if we can characterize the purely tree-colorable graphs, then the uniquely 4-colorable maximal planar graph conjecture is proved naturally. So, we will study these three kinds of maximal planar graphs in the following series of papers, and mainly consider the purely tree-colorable graphs in this section.

3.1 Purely tree-colorable maximal planar graph Conjecture with minimum degree 5

In Ref. [41], it has been found that the icosahedron is a purely tree-colorable maximal planar graph with minimum degree 5, which has ten different tree-colorings shown in Fig. 2.

Let G be a planar graph. If there exists a $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(u)=v$ for any two vertices $u,v \in V(G)$, then G is said to be vertex transitive. Obviously, the icosahedron is vertex transitive.



(a) f_1:12-cycle　(b) f_2:12-cycle　(c) f_3:tree-coloring　(d) f_4:tree-coloring

Fig. 1　All 4-colorings of a maximal planar graph with order 11

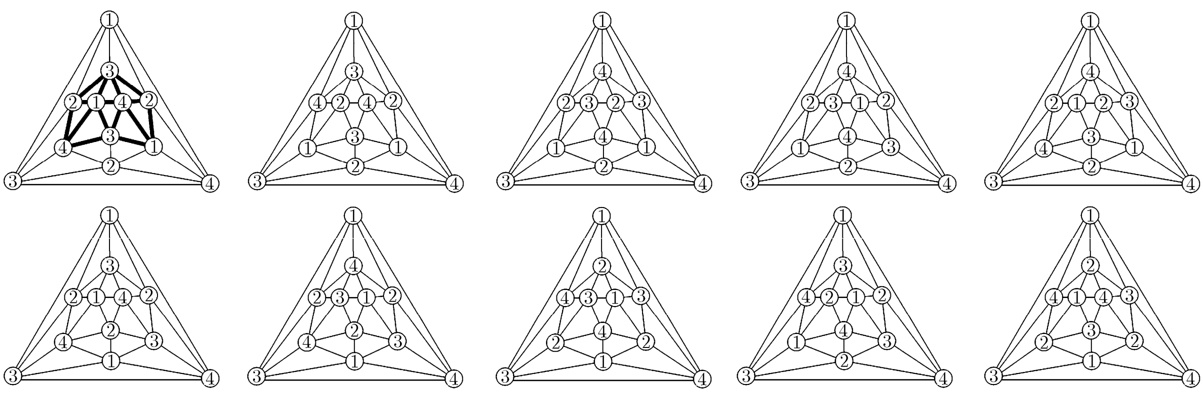


Fig. 2　The icosahedron and its all ten 4-colorings

Therefore, the domino configuration contained in the icosahedron with 2 inner vertices must be unique. We can choose the bold domino configuration of the first graph in Fig.2 as a representative.

Conjecture 4 Let G be a maximal planar graph with minimal degree 5. Then G is purely tree-colorable if and only if G is the icosahedron.

3.2 Dumbbell-maximal planar graphs

3.2.1 Dumbbell transformation

The object graph of dumbbell transformation is a closed dumbbell, namely a 4-wheel, shown in Fig. 3. The so called dumbbell transformation is to replace the closed dumbbell by the fourth graph shown in Fig. 3(a) or Fig. 3(b). The specific steps are as follows:

Step 1 Split the wheel center v_2 into two vertices v_2 and v_2', vertically or horizontally, shown in the second graph of Fig. 3(a) or Fig. 3(b), respectively.

Step 2 Extend to the third graph of Fig. 3(a) or Fig. 3(b).

Step 3 Add a 2-length path in the 6-cycle $v_1v_3v_2'v_5v_4v_2v_1$, vertically or horizontally, and then join edges between the 2-length path and 6-cycle to form the configuration shown in the fourth graph of Fig. 3(a) or Fig. 3(b).

Moreover, the inverse operation of dumbbell transformation is called dumbbell contracting-transformation, and the fourth graph in Fig. 3(a) is called the object graph of dumbbell contracting-transformation. In Fact, the dumbbell transformation is equivalent to the extending 334-wheel operation, as shown in Fig. 3(c). For more information, readers can refer to this series of article (2) (see Ref. [39]).

3.2.2 Structure of dumbbell-maximal planar graphs

Let J^{4k+1} be a maximal planar graph with order $4k+1$ and minimum degree 4, and contain a dumbbell contracting-transformation object subgraph. If the graph obtained by implementing the dumbbell contracting-transformation still contains such an object subgraph, then implement the dumbbell contracting-transformation again. Continue in this way, we call J^{4k+1} a dumbbell-maximal planar graph if the finally obtained graph is J^9. According to this definition, the graph obtained by implementing the dumbbell transformation from a dumbbell-maximal planar graph is also a dumbbell- maximal planar graph.

In the following, we give the steps to construct a dumbbell-maximal planar graph of order $4k+5$ from $J^{4k+1}\,(k \geq 2)$.

Step 1 Find all unidentical closed dumbbells

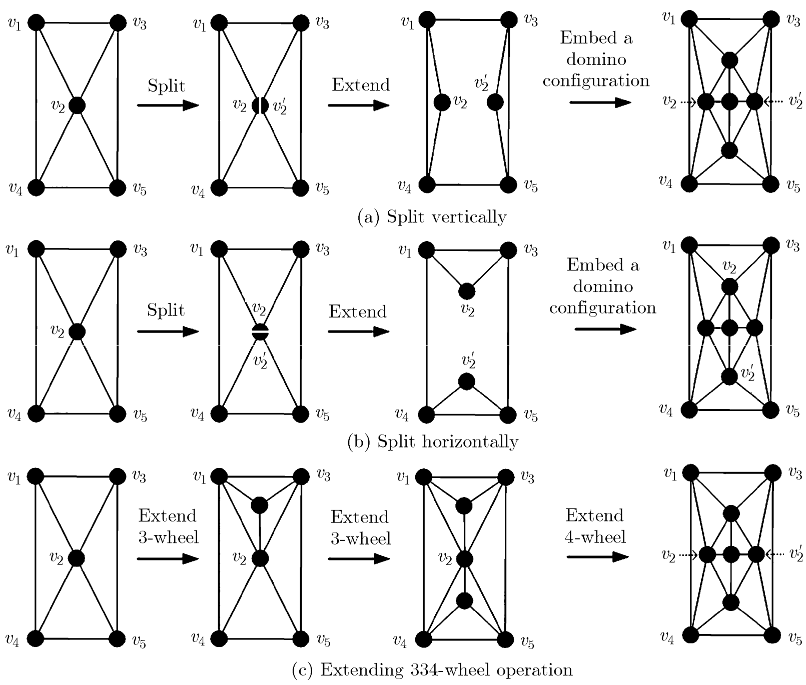


Fig. 3 The object graph and the process of dumbbell transformation

in J^{4k+1} (that is, find all unidentical 4-wheels). As shown in Fig. 4(a), 4(b), 4(c), 4(d), the dumbbell-maximal planar graphs with orders 9, 13, 17, 17 contain three 4-wheels respectively, while the numbers of unidentical 4-wheels are 1, 2, 2, 2;

Step 2 Implement the dumbbell transformation on each unidentical 4-wheel in J^{4k+1} , then we can obtain dumbbell-maximal planar graphs with order $4k+5$. For example, J^{13} (shown in Fig. 4(b)) is obtained by implementing the dumbbell transformation on the 4-wheel bolded in J^9 (shown in Fig. 4(a)). It is easy to prove that there are two closed dumbbells in J^{13} which are identical. So, we obtain two dumbbell-maximal planar graphs with order 17 by implementing the dumbbell transformation, shown in Fig. 4(c) and Fig. 4(d), respectively.

3.2.3 Properties of dumbbell-maximal planar graphs

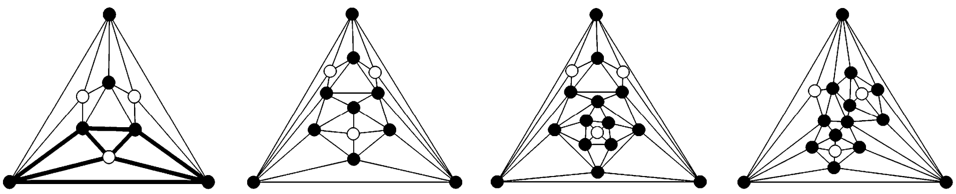
We further discuss some properties of dumbbell maximal planar graphs in this subsection.

Theorem 1 (1) Any dumbbell-maximal planar graph contains exactly 3 vertices of degree 4; (2) Any dumbbell-maximal planar graph has $4k+1$ vertices, where $k \geq 2$; (3) Every dumbbell-maximal planar graph is purely tree-colorable, and each dumbbell maximal planar graph with order $4k+1$ has exactly 2^{k-1} different colorings.

Proof (1) and (2) Note that J^9 has 3 identical vertices of degree 4. According to the definition of dumbbell transformation, there exist exactly 3 vertices of degree 4 in the dumbbell-maximal planar graph J^{13}, which is obtained from J^9 by a dumbbell transformation. Continue in this way, it is known that for any $k \geq 2$, J^{4k+1} contains exactly 3 vertices of degree 4 and each dumbbell-maximal planar graph has $4k+1$ vertices.

(3) By induction. Obviously, the result holds if $k=2$. The dumbbell-maximal planar graph of order 13 has totally 4 different colorings and each of these colorings is tree-coloring, shown in Fig. 5. So, the theorem holds.

Suppose that the result holds for $k \geq 3$, that is, J^{4k+1} is purely tree-colorable and has exactly 2^{k-1} different colorings. Now, we discuss the case of $k+1$. Let $W_4 = v_2 - v_1 v_4 v_5 v_3$ be a 4-wheel of J^{4k+1} and v_2 be the center of the wheel, shown in Figs. 6(b). Implement the dumbbell transformation on dumbbell $X_1 = \Delta v_1 v_2 v_3 \cup \Delta v_2 v_4 v_5$ of W_4 (shown in Figs. 6(a) and 6(c)). Suppose that the obtained graph $\zeta_{334}^{+}(J^{4k+1})$ contains a bicolored cycle, then there must exist a $f \in C_4^0(\zeta_{334}^{+}(J^{4k+1}))$, such that f contains a bicolored cycle. So, there are two cases: one is $f(v_2^{'}) = f(v_2)$, shown in Fig. 6(a); the other is $f(x_1) = f(x_2)$, shown in Fig. 6(c). For both of the two cases, we can obtain J^{4k+1} by implementing the dumbbell contracting-transformation. Since the dumbbell contracting-transformation is indeed implementing contract 4-wheel operation one time and then contract 3-wheel two times, the natural coloring (based on f) of the obtained graph contain bicolored cycles. This contradicts the fact that J^{4k+1} is purely tree-colorable. So, $\zeta_{334}^{+}(J^{4k+1})$



(a) Order 9 (b) Order 13 (c) Order 17 (d) Order 17

Fig. 4 Four dumbbell-maximal planar graphs with lower orders

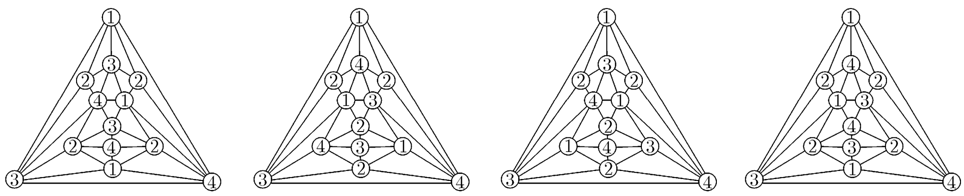
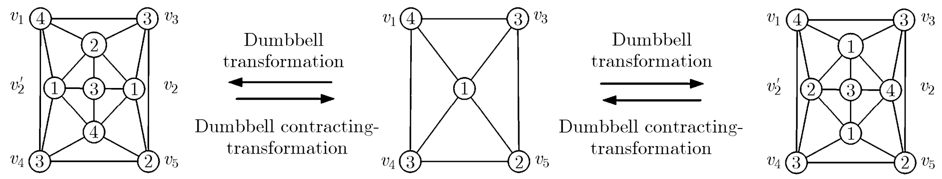


Fig. 5 All 4-colorings of the dumbbell maximal planar graph with order 13



(a) Split vertically (b) A closed dumbbell (c) Split horizontally

Fig. 6 Dumbbell transformation and dumbbell contracting-transformation based on colorings

is purely tree-colorable.

Without loss of generality, we suppose that each 4-wheel of J^{4k+1} is colored as Fig. 6(b). Then subject to the coloring of the external cycle, the dumbbell contracting-transformation object subgraph of $\zeta_{334}^{+}(J^{4k+1})$ has exactly 2 colorings, as shown in Fig. 6(a) and Fig. 6(c), respectively. So, $\zeta_{334}^{+}(J^{4k+1})$ has exactly $2^{k-1}\times 2=2^{k}$ different colorings. The proof is completed.

3.2.4 Enumeration

J^9 is the unique dumbbell maximal planar graph with order 9. Since the three 4-wheels in J^9 are identical, the dumbbell maximal planar graph with order 13 is also unique. Recall that two closed dumbbells in J^{13} are identical. So, there are two dumbbell-maximal planar graphs with order 17, shown in Fig. 4(c), 4(d). More generally, we have:

Theorem 2 Let t_k denote the number of dumbbell maximal planar graphs with order $4k+9$, where $k\geq 0$. Then

$$t_k=\frac{(k+3)^2}{12}-\frac{7}{72}+\frac{(-1)^k}{8}+\frac{2}{9}\cos\frac{2k\pi}{3}$$

The proof can be found in Ref. [41].

4 Recursive Maximal Planar Graphs

In addition to the Four-Color Conjecture, the uniquely 4-colorable maximal planar graph conjecture (Conjecture 3) has been a very influential conjecture in the graph coloring theory, which has 43 years of history. Uniquely 4-colorable maximal planar graphs are conjected to be recursive maximal planar graphs[28], so we study on this class of graphs in this section.

The so called recursive maximal planar graph (RMPG) is obtained from K_4 by embedding a 3-degree vertex in some triangular face continuously, where embedding a 3-degree vertex in a triangular is to add a vertex on the face of this triangular, and then connect this vertex to the three vertices of the triangular. In this paper, Λ denotes the set consisting of all RMPGs and Λ_n the set of graphs in Λ with order n. Let $\gamma_n=|\Lambda_n|$. Obviously, $\gamma_n=1$, $n=4,5,6$, the corresponding RMPGs are shown in Fig. 7.

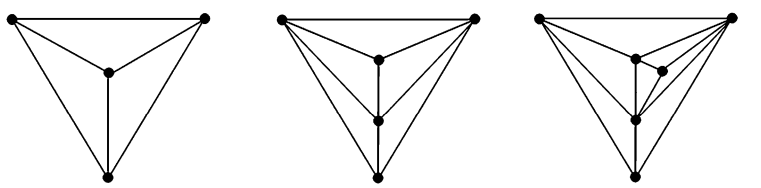
4.1 Basic properties

Theorem 3 Let G be an RMPG with order n. Then G has at least two vertices of degree 3. When $n\geq 5$, any two vertices of degree 3 are not adjacent to each other.

Proof By induction on the number n of vertices. When $n=4,5,6$, $\gamma_4=\gamma_5=\gamma_6=1$, and the corresponding graphs are shown in Fig. 7. So the result is true obviously.

Assume that the theorem holds when $n\geq 6$. That is, for any RMPG G with n vertices, it has at least two 3-degree vertices, and all the vertices of 3-degree are not adjacent to each other.

A graph $G\in\Lambda_{n+1}$ is constructed by embedding a 3-degree vertex v in any triangular face of an RMPG with n vertices, assuming $N_G(v)=\{v_1,v_2,v_3\}$. By induction, there are at least two 3-degree vertices, and all the vertices of



(a) An RMPG with order 4 (b) An RMPG with order 5 (c) An RMPG with order 6

Fig. 7 Three RMPGs with orders 4,5,6

3-degree are not adjacent to each other in $G-v$. For $G-v$, if 3-degree vertices are included in $\{v_1,v_2,v_3\}$, then one exists at most, saying v_1. Obviously, there is at least another 3-degree vertex except for v_1 in $G-v$, and all those 3-degree vertices are not adjacent to each other. Since v is a 3-degree vertex of G and it is not adjacent to any other vertices except for $\{v_1,v_2,v_3\}$, there are also at least two 3-degree vertices in G, and all those 3-degree vertices are not adjacent to each other. Thus, the conclusion holds. For $G-v$, if there exists no 3-degree vertices in $\{v_1,v_2,v_3\}$, the conclusion holds by the same way above. The theorem follows by the principle of induction.

Theorem 4 (1) There exists no maximal planar graph having exactly two adjacent vertices of degree 3; (2) There exists no maximal planar graph with exactly three vertices of degree 3 adjacent to each other.

Proof (1) By contradiction. Assume that G is a maximal planar graph with two adjacent vertices $u,v\in V(G)$, satisfying $d(u)=d(v)=3$. Let $N(u)=\{v,x,y\}$. Notice that G is a maximal planar graph and u must be in a triangular face which consists of the vertices v, x, and y. In other words, v is adjacent to x and y. These four vertices can form a subgraph K_4, as shown in Fig. 8. Since G is a maximal planar graph, if there exist any other vertices of G, then it can form a triangle with u or v. It contradicts $d(u)=d(v)=3$. Otherwise, if there exists no other vertices, then G is isomorphism to K_4, which has four vertices of degree 3. Therefore, there exists no maximal planar graph with two adjacent vertices of 3-degree exactly. Similarly, (2) holds.

Theorem 5 If G is a maximal planar graph with only one vertex of degree 3, then a graph without any 3-degree vertex can be obtained from G by deleting 3-degree vertices repeatedly.

Proof Let v be the unique vertex of degree 3 in G, and $N_G(v)=\{u_1,u_2,u_3\}$. Thus, these three vertices can form a triangle. Let $G_1=G-v$. Obviously, G_1 is a maximal planar graph. There exist four possible cases in G_1 as follows:

(1) $\delta(G_1)\ge 4$;

(2)There exists only one 3-degree vertex;

(3)There exactly exist two 3-degree vertices;

(4)There exactly exist three 3-degree vertices.

For Case (1), the theorem holds naturally, and the Cases (3) and (4) do not exist by Theorem 4. So we just need to consider the Case (2). In this case, there exists only one 3-degree vertex in subgraph G_1, denoted by u_1. Let $G_2=G_1-u_1$. Like the method mentioned above, if $\delta(G_2)\ge 4$, then the theorem holds. Otherwise, G_2 must contain a 3-degree vertex. In this way, we can get $\delta(G_m)\ge 4$ within finite m steps. Otherwise, $G_m\cong K_4$ when G_m contains only four vertices. It means that the graph G is an RMPG. But there is only one 3-degree vertex in G, which contradicts Theorem 3.

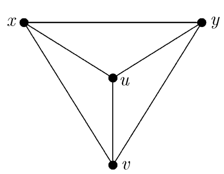


Fig. 8　K_4

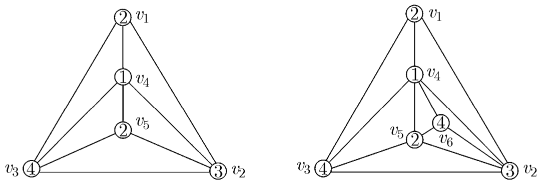
Theorem 6 Let G be an RMPG with order ≥ 5 and v be a 3-degree vertex in G. Then $G-v$ is still an RMPG.

Proof Let G be the RMPG obtained from $K_3=v_1v_2v_3$ by adding 3-degree vertices v_4, $v_5,\cdots,v_n$ successively. If $v\in\{v_4,v_5,\cdots,v_n\}$, $G-v$ is obviously an RMPG when $v=v_n$; when $v=v_i$, $4\le i\le n-1$, delete 3-degree vertices $v_n,v_{n-1},\cdots,v_{i+1}$ successively in $G-v$, the resulting graph denoted by G'. It is easy to prove that G' can be obtained from G by deleting 3-degree vertices $v_n,v_{n-1},\cdots,v_{i+1},v_i$ successively, and G' is an RMPG. So, $G-v$ is still an RMPG. If $v\in\{v_1,v_2,v_3\}$, assuming $v=v_1$, then $G-v$ is an RMPG obtained from the triangle $\Delta v_2v_3v_4$ by adding 3-degree vertices $v_5,v_6,\cdots,v_n$ successively.

4.2 (2,2)-recursive maximal planar graphs

In this subsection, we introduce and study (2,2)- RMPGs, which are a special class of RMPGs. An RMPG is called a (2,2)-RMPG if it contains only two vertices of degree 3, and the distance between them is 2. It is easy to check that there exists only one (2, 2)-RMPG with orders 5 and 6, shown in Fig. 9(a) and Fig. 9(b), respectively.

To understand the structure of (2,2)-RMPGs,



(a) A (2,2)-RMPG with order 5 (b) A (2,2)-RMPG with order 6

Fig. 9 (2,2)-RMPGs with orders 5 and 6

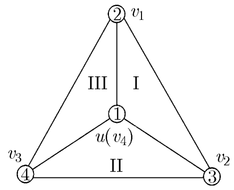


Fig. 10 The basic frame diagram of the color-coordinate system

we divide three inner faces of the complete graph K_4 into three regions, and label its vertices correspondingly. In Fig.10, the triangle labeled by v_1, v_2, v_3 is called the outside triangle, and the vertex u is called the central vertex. Here we define that the vertices v_1, v_2, v_3, v_4 are colored with color 2, color 3, color 4, and color 1, respectively. The four vertices and their corresponding colorings are called the basic axes in the color-coordinate system of a (2,2)-RMPG. Four color axes are u (color 1), v_1 (color 2), v_2 (color 3), v_3 (color 4). Obviously, there exists no (2,2)-RMPG with order 4; and there is only one (2,2)-RMPG with 5 vertices up to isomorphism, which can be obtained by embedding a 3-degree vertex in the region I, II or III of the graph K_4 (shown in Fig.10). Without loss of generality, we make an agreement that new vertices are only added in the region II. Thus, the vertex is colored by color 2 (Fig. 9(a)); the non-isomorphic (2,2)-RMPGs with order 6 can be obtained by embedding a 3-degree vertex in any region of the (2,2)-RMPG with order 5. It is easy to prove that this kind of graphs with 6 vertices obtained by embedding a new vertex in any face are isomorphic. Thus, there is only one (2,2)-RMPGs with order 6. In general, we make an agreement that the 6th vertex is embedded in the face composed of the vertices v_2, v_4, v_5 (*i.e.* the sub-region I of the region II), which is colored by color 4 (Fig. 9(b)). Further, for (2,2)-RMPGs with higher orders, we restrict that new vertices are only added in the regions I and II, but not in the region III.

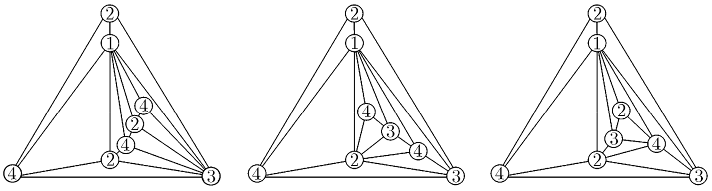
Based on the agreement above, we can discuss the classification of (2,2)-RMPGs. Two methods for this purpose will be introduced as follows.

The first is based on the region where the 3-degree vertices are embedded: (1) (2,2)-RMPGs which are obtained by successively embedding 3-degree vertices only in the region II, such graphs are shown in Fig.11; (2) (2,2)-RMPGs are obtained by successively and randomly embedding 3-degree vertices in the regions I and II, shown in Fig.12. For maximal planar graphs, we have a straightforward fact as follows.

Proposition 1 Any face in a maximal planar graph can become the infinite outside face.

That is, the (2,2)-RMPGs mentioned above are obtained by embedding 3-degree vertices randomly in the regions I and II. We can transform any 3-degree vertex in the regions I or II to the outside triangular face by Proposition 1, which is equivalent to the first classification. It means that this kind of (2,2)-RMPGs are obtained by successively embedding 3-degree vertices only in the region II. Therefore, we only consider this kind of graphs in the later sections.

The second method is based on whether there exists a common edge between the two triangular surfaces of two 3-degree vertices or not. It is called the adjacent type if there is a common edge;



(a) The adjacent type (b) The non-adjacent type (c) The non-adjacent type

Fig. 11 The (2,2)-RMPGs obtained by embedding 3-degree vertices only in the region II

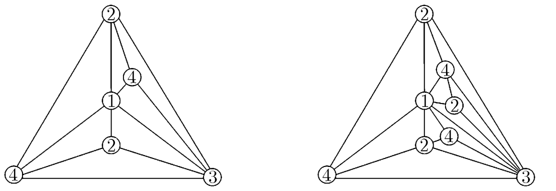


Fig. 12 The (2,2)-RMPGs obtained by embedding 3-degree vertices in the regions I and II randomly

otherwise, the non-adjacent type. As shown in Fig. 11, the first graph belongs to the adjacent type, whereas the last two graphs belong to the non-adjacent type.

In Fig. 9(a), the (2, 2)-RMPG with order 5 is a double-center wheel, and the degree of vertices in the neighbor of each center is 4, where a double-center wheel is a specific maximal planar graph constructed by the join of a cycle and two single vertices. When the order of a (2,2)-RMPG is not less than 6, we have the following results.

Theorem 7 (1) Let G be a (2,2)-RMPG with order $n(\geq 6)$. Then for each 3-degree vertex v in G, there only exists one vertex with order 4 in $N(v)$. (2) Every (2,2)-RMPG G of non-adjacent type with order n (≥ 7) has only one $(n-1)$-degree vertex, and it is called the central vertex of the graph G, denoted by u. Furthermore, in any partitions of color class in G, only the central vertex is colored with color 1. (3) For the (2,2)-RMPGs of adjacent type obtained by embedding the 3-degree vertices only in the region II, only its central vertex is colored with color 1 and also only its color axis v_2 is colored with color 3.

Proof (1) By induction. There is only one maximal planar graph of order 5 (shown in Fig. 9(a)), it is also a double-center wheel, so all triangular faces are equivalent. Therefore, in the view of isomorphism, there exist only one RMPG with order 6, also a (2,2)-RMPG (shown in Fig. 9(b)). Thus, the theorem holds when $n=6$.

Assume that the theorem holds when $n(\geq 6)$. We now consider a (2, 2)-RMPG G of order $n+1$. Suppose that v is a 3-degree vertex in G, then there are two cases as follows:

Case 1 two or three vertices of 4-degree are included in $N(v)$, then $G-v$ is also an RMPG with order at least 6 which contains two or three vertices with degree 3 adjacent to each other, which contradicts with Theorem 3.

Case 2 There exists no vertex of degree 4 in $N(v)$, then $G-v$ is also an RMPG with order at least 6 which contains only one vertex of degree 3, which contradicts with Theorem 3.

In conclusion, we have proved that only one vertex of degree 4 is included in $N(v)$.

(2) and (3) can be proved by the gradual construction of (2,2)-RMPGs. The proof is completed.

Corollary 1 Let G be a (2,2)-RMPG with order at least 6. Then for any 3-degree vertex v in G, $G-v$ is still a (2,2)-RMPG.

Theorem 8 Let η_n denote the number of the non-isomorphic (2,2)-RMPG with order $n(\geq 6)$, then $\eta_n \leq 2^{n-6}$. In particular, $\eta_6=1, \eta_7=2, \eta_8=3, \eta_9=6$.

Proof Any (2,2)-RMPG can be obtained by embedding 3-degree vertices in the region II from the (2,2)-RMPG with order 6. When embedding the $(i+1)$-th vertex in (2,2)-RMPG with order i, $6 \leq i \leq n-1$, there are only 2 triangles can be chosen for the non-adjacent type and 3 triangles for the adjacent type. It is easy to prove that the graphs obtained by embedding 3-degree vertices in the two faces incident with the 4-degree vertex are isomorphic. Therefore, $\eta_n \leq 2^{n-6}$. The (2,2)-RMPGs with orders 7, 8, 9 are shown in Fig. 13, respectively. The proof is completed.

4.3 The colorings of extending 4-wheel operation graphs

Without loss of generality, we can always assume that the (2,2)-RMPG G is obtained by embedding 3-degree vertices only in the region II in following discussion. Thus, a (2,2)-RMPG G can be uniquely represented by its color sequence.

Let $V(G)=\{v_1,v_2,v_3,v_4=u,v_5,\cdots,v_n\}$, where vertex $v_1=x$ indicates the first fixed vertex of degree 3, vertex $v_n=y$ indicates the second vertex of degree 3; the vertices $v_1=x$, v_2, v_3 and $v_4=u$ indicate the 1st, 2nd, 3rd, 4th color axis respectively; while the vertex $v_4=u$ is the central vertex; the vertex v_{n-1} denotes the 3-degree vertex of the subgraph $G_{n-1}=G-v_n$; the vertex v_{n-2} denotes the 3-degree vertex of the subgraph G_{n-2}

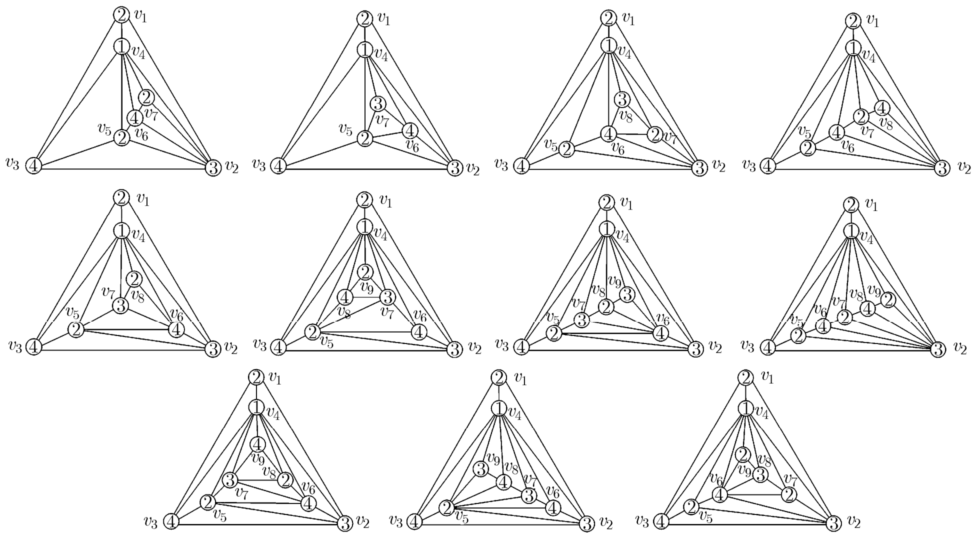


Fig. 13 All of the (2,2)-RMPGs with orders 7, 8, 9

$=G_{n-1}-v_{n-1}$; the rest can be deduced in the same way. The sequence $c_1c_2\cdots c_n$ is used to indicate the corresponding color sequence of the vertex sequence $v_1,v_2,\cdots,v_n$, where c_i is the color of v_i in the (2,2)-RMPG G, $c_i\in\{1,2,3,4\}$.

According to the definition of (2,2)-RMPGs, this representation also determines the structure of a graph. This structure starts from K_4 (shown in Fig. 10), and selects a triangular face to embed vertices according to the color of each vertex.

For example, for the color sequence $c_1c_2c_3c_4c_5c_6c_7c_8c_9=234124323$, its corresponding (2, 2)-RMPG is the middle graph of row 3 in Fig. 13.

For the color sequence of a (2,2)-RMPG, we can obtain the following:

Theorem 9 Let $c_1c_2\cdots c_n$ be the color sequence of a (2,2)-RMPG G. With the agreement "embedding 3-degree vertices only in the region II", the colors of the first six vertices in this sequence are determined, namely $c_1=2$, $c_2=3$, $c_3=4$, $c_4=1$, $c_5=2$, $c_6=4$; if G belongs to the adjacent type, then $c_7=2$; otherwise, $c_7=3$ or 2.

Proof It follows from Fig. 10 that $c_1=2$, $c_2=3$, $c_3=4$ and $c_4=1$. Then we obtain $c_5=2$ by Fig. 9(a) and $c_6=4$ by Fig. 9(b). Again by Fig. 9(b), $c_7=3$ or 2. The proof is completed.

In the following, we are devoted to discussing the vertex coloring problem of the induced graph from a (2,2)-RMPG by extending 4-wheel operation. We know that any given (2,2)-RMPG is uniquely 4-colorable, and every vertex can also be colored determinately.

Let G be a (2,2)-RMPG, f be the unique 4-coloring of G, and xuy be a path of length 2 in G. Obviously, there exists a coloring f^* of the graph $\zeta_4^+(G)$ which is induced from G by extending 4-wheel operation on the path xuy, and

$$f^*(z)=\begin{cases}f(u), & z=u',\, or\, u\\ \{1,2,3,4\}\setminus\{f(x),f(u),f(y)\}, & z=v\\ f(z), & \text{otherwise}\end{cases}$$

where v is the new added vertex of the 4-wheel. Namely, f^* is a coloring of graph $\zeta_4^+(G)$. Vertices u and u' are assigned the same color under f^*, and the new added vertex v is assigned the color different from vertices x,u,y, while the colors of the rest vertices remain unchanged. We refer to f^* as the natural 4-coloring of the graph $\zeta_4^+(G)$.

Naturally, one question is proposed that whether the graph $\zeta_4^+(G)$ obtained by extending 4-wheel operation is uniquely 4-colorable or not. In fact, the answer is negative, that is, $|C_4^0(\zeta_4^+(G))|\ge 2$.

Theorem 10 Let G be a (2,2)-RMPG with order n, f be the unique 4-coloring of G, x, y be two vertices of degree 3 and u be the central vertex of G. Then the graph $\zeta_4^+(G)$, obtained from G by extending 4-wheel operation on the path xuy, is not uniquely 4-colorable.

Proof Obviously, the conclusion holds when $f(x)=f(y)$. So we only need to consider the case when $f(x)\neq f(y)$. Let $N(x)=\{u,x_1,x_2\}$, $f(u)=1$, $f(x)=2$, $f(x_1)=3$, $f(x_2)=4$. According to Theorem 7, only vertex u in G can be colored with coloring 1, so the vertex y should be colored with coloring 3 or 4. Without loss of generality, let $f(y)=f(x_1)=3$. Since G is a (2,2)-RMPG, it can be obtained from $K_4=G[\{x,u,x_1,x_2\}]$ by adding $n-4$ vertices of degree 3 in turn, namely v_1, $v_2,\cdots,v_{n-4}$. Let w be the vertex adjacent to x_1 and with the largest subscript in $v_1,v_2,\cdots,v_{n-4}$, then $f(w)=2$ or 4.

In $\zeta_4^+(G)$, let v and u' be the new added vertices, where v is the center of the 4-wheel obtained. $N(v)=\{x,y,u,u'\}$, where u is adjacent to x_1, u' is adjacent to x_2. By the definition of natural coloring, $f^*(w)=f(w)=2$ or 4.

If $f^*(w)=2$, then remove the color of vertices $x,x_1,v,$ and exchange the colors of the 14-component(namely, some component of the graph induced by all vertices colored 1 and 4) containing u, and then color the vertices x,x_1,v with 3,1,2. Obviously, we obtain a 4-coloring of $\zeta_4^+(G)$ which is different from f^*. If $f^*(w)=4$, then remove the color of vertices x,x_1, and exchange the colors of the 12-component containing u, and then color the vertices x,x_1 with 3,1. Also, the obtained coloring is a 4-coloring of $\zeta_4^+(G)$ different from f^*. The proof is completed.

Remark In Theorem 10, we showed that the graph $\zeta_4^+(G)$ obtained from a (2,2)-RMPG G by extending 4-wheel operation on the path xuy is not uniquely 4-colorable, where x,y must be 3-degree vertices of G. When x,y are not vertices of degree 3, the graph $\zeta_4^+(G)$ may be uniquely 4-colorable.

5 An Idea to Prove the Uniquely 4-colorable Maximal Planar Graph Conjecture

The uniquely 4-colorable maximal planar graph conjecture is still an open problem until now. The object of this conjecture is the RMPG, so we discussed the properties of RMPGs in detail. Also, we have proposed the purely tree-colorable maximal planar graph conjecture in Ref. [41], and if this conjecture holds, then the uniquely 4- colorable maximal planar graph conjecture is solved. Specially, we studied the dumbbell maximal planar graphs in Section 3 of this article. In this section, we give an idea to prove the uniquely 4-colorable maximal planar graph conjecture, which in fact is an idea to prove the purely tree-colorable maximal planar graph conjecture.

Let G be a purely tree-colorable maximal planar graph, W_4, W_5 be 4-wheel and 5-wheel of G, respectively. Moreover, we use τ_i, $i=1,2,3$ and τ_1' denote the four generating operators of the domino configuration (see Ref. [39]), and $\tau_i(W_4)$ denotes the graph obtained by implementing τ_i operation on a 4-wheel W_4 of G. Other symbols can be defined similarly.

In the following, we give an idea to prove the purely tree-colorable maximal planar graph conjecture. Specifically, we need to prove the 9 cases below one by one.

Case 1 If G is a purely tree-colorable maximal planar graph, then $\tau_1(W_4)$ is cycle-colorable.

Case 2 If G is a purely tree-colorable maximal planar graph, then $\tau_1(W_5)$, $\tau_2(W_5)$ are cycle-colorable.

Case 3 If G is a purely tree-colorable maximal planar graph, then $\tau_2(W_5)$ is cycle-colorable.

Case 4 If G is a purely tree-colorable maximal planar graph, then $\tau_3(W_5)$ is cycle-colorable.

Case 5 If G is a purely tree-colorable maximal planar graph, then $\tau_1(\tau_1(W_4))$ is cycle-colorable.

Case 6 If G is a purely tree-colorable maximal planar graph, then $\tau_2(\tau_1(W_4))$ and $\tau_1(\tau_2(W_4))$ are cycle-colorable.

Case 7 If G is a purely tree-colorable maximal planar graph, then $\tau_1'(\tau_2(W_4))$ is cycle-colorable.

Case 8 If G is a purely tree-colorable maximal planar graph, then $\tau_1'(\tau_2(W_5))$ is cycle-colorable.

Case 9 If G is a purely tree-colorable maximal planar graph, then $\tau_2(\tau_2(W_4))$ is purely tree-colorable.

For more information, we refer the reader to XU[39] (Subsection 3.4, especially for the Fig. 17). In addition, studies on this conjecture will be given in the later article of this series.

6 Conclusion and Prospection

In this paper, we mainly studied a famous conjecture in graph coloring theory, the uniquely 4-colorable maximal planar graph conjecture. Since any uniquely 4-colorable maximal planar graph is conjected to be an RMPG, we studied the RMPGs in Section 4.

In Ref. [41], we proposed the purely tree-colorable graphs conjecture, which states that a maximal planar graph is purely tree-colorable if and only if it is the icosahedron or a dumbbell maximal planar graph. Moreover, we proved that if this conjecture holds, then the uniquely 4-colorable maximal planar graph conjecture also holds. So, we further studied the structures and properties of dumbbell maximal planar graphs in Section 3.

Finally, we give an idea to prove the purely tree- colorable graphs conjecture, which naturally implies the uniquely 4-colorable maximal planar graph conjecture. Specifically, suppose that G is purely tree-colorable, there are 9 cases to prove the conjecture based on the extending-contracting operation proposed in this series of article (2) and the result that any maximal planar graph with order ≥ 9 and minimum degree ≥ 4 has a father-graph or a grandfather-graph. Among the 9 cases, 8 cases are negative and only one is positive.

The work of this paper lays a foundation for proving the purely tree-colorable graphs conjecture. In the later article of this series, we will give the extending-contracting operational system of 4-colored maximal planar graphs, simply as the colored extending-contracting operational system. On this basis, the purely tree-colorable graphs conjecture is expected to be solved.

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