极大平面图的结构与着色理论

(1)色多项式递推公式与四色猜想

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摘　要：该文给出了极大平面图 G 的色多项式递推计算公式：若 $\delta(G)=4$，W_4^v 是 G 中轮心为 v，轮圈为 $v_1v_2v_3v_4v_1$ 的 4-轮，则 $f(G,4)=f(G_1,4)+f(G_2,4)$，其中 $G_1=(G-v)\circ\{v_1,v_3\}$，$G_2=(G-v)\circ\{v_2,v_4\}$；若 $\delta(G)=5$，W_5^v 是 G 中 v 为轮心，以 $v_1v_2v_3v_4v_5v_1$ 为轮圈的 5-轮，则 $f(G,4)=[f(G_1,4)-f(G_1\cup\{v_1v_4,v_1v_3\},4)]+[f(G_2,4)-f(G_2\cup\{v_3v_1,v_3v_5\},4)]+[f(G_3,4)-f(G_3\cup\{v_1v_4\},4)]$，其中 $G_1=(G-v)\circ\{v_2,v_5\}$，$G_2=(G-v)\circ\{v_2,v_4\}$，$G_3=(G-v)\circ\{v_3,v_5\}$，“$\circ$”表示收缩运算；进而讨论了使用公式证明四色猜想的应用：将四色猜想转化成研究一种特殊图类：4-色漏斗型伪唯一 4-色极大平面图。

关键词：四色猜想；极大平面图；色多项式；伪唯一 4-色平面图；4-色漏斗

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Theory on the Structure and Coloring of Maximal Planar Graphs

(1)Recursion Formulae of Chromatic Polynomial and Four-Color Conjecture

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Abstract: In this paper, two recursion formulae of chromatic polynomial of a maximal planar graph G are obtained: when $\delta(G)=4$, let W_4^v be a 4-wheel of G with wheel-center v and wheel-cycle $v_1v_2v_3v_4v_1$, then $f(G,4)=f((G-v)\circ\{v_1,v_3\},4)+f((G-v)\circ\{v_2,v_4\},4)$; when $\delta(G)=5$, let W_5^v a 5-wheel of G with wheel-center v and wheel-cycle $v_1v_2v_3v_4v_5v_1$, then $f(G,4)=[f(G_1,4)-f(G_1\cup\{v_1v_4,v_1v_3\},4)]+[f(G_2,4)-f(G_2\cup\{v_3v_1,v_3v_5\},4)]+[f(G_3,4)-f(G_3\cup\{v_1v_4\},4)]$, $G_1=(G-v)\circ\{v_2,v_5\}$, $G_2=(G-v)\circ\{v_2,v_4\}$, $G_3=(G-v)\circ\{v_3,v_5\}$, where“$\circ$”denotes the operation of vertex contraction. Moreover, the application of the above formulae to the proof of Four-Color Conjecture is investigated. By using these formulae, the proof of Four-Color Conjecture boils down to the study on a special class of graphs, viz., 4-chromatic-funnel pseudo uniquely-4-colorable maximal planar graphs.

Key words: Four-Color Conjecture; Maximal planar graphs; Chromatic polynomial; Pseudo uniquely-4-colorable planar graphs; 4-chromatic-funnel

1　引言

本文所言之图皆指有限简单无向图。对于给定图 G，分别用 $V(G)$，$E(G)$，$d_G(v)$ 和 $N_G(v)$ 来表示图 G 的顶点集，边集，顶点 v 的度数和顶点 v 的邻域 (即与顶点 v 相邻的所有顶点构成的集合)，可分别简记为 V，E，$d(v)$，$N(v)$。图 G 的**阶**是 $V(G)$ 中元素的个数 $|V(G)|$。若 $V'\subseteq V$，$E'\subseteq E$，且 E' 中每条边的两个端点均在 V' 中，则称图 $H=(V',E')$ 是图 G 的一个**子图**。在子图 H 中，如果对于 $\forall u,v\in V(H)$，u,v 在 G 中相邻当且仅当它们在图 H 中相邻，则称 H 为 G 的一个由 V' **导出的子图**，记为 $G[V']$。对于点不交的两个图 G，H，若将图 G 中的每个顶点与图 H 中的每个顶点相连边，则得到的新图称为图 G 与图 H 的**联图**，记为 $G\vee H$。用 K_n 表示 n-阶完全图。平凡图 K_1 与 n 阶圈 C_n 的联图 $C_n\vee K_1$ 称作**轮图** W_n (图 1 分别列出了轮图 W_3,W_4,W_5)，其中 C_n 称为该轮的**圈**；K_1 的顶点称为该轮的**轮心**。若 $K_1=\{x\}$，有时把轮图 W_n 的圈 C_n 也用 C^x 来表示。

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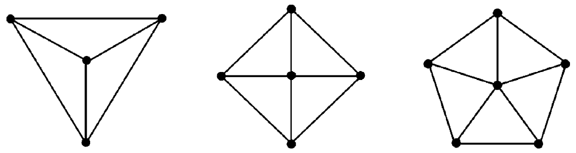


(a) W_3 (b) W_4 (c) W_5

图 1 轮图示例

若图 G 的所有顶点的度数都为 k，则称图 G 是 k-**正则图**。3-正则图也称为**立方图**。

图 G 的一个**顶点着色** f 是指对图 G 中的每个顶点分配一种颜色，且满足 G 中每条边的两个端点分配不同的颜色。图 G 的一个**正常** k-**顶点着色**，简称为 k-**顶点着色**或 k-**着色**，是指从图 G 的顶点集 V 到颜色集 $C(k)=\{1,2,\cdots,k\}$ 的一个映射 f，满足对任意的 $xy\in E(G)$，有 $f(x)\neq f(y)$。如果在 G 中存在一个正常 k-顶点着色，则称图 G 是 k-**可着色的**。图 G 的**色数**，记作 $\chi(G)$，是指满足图 G 为 k-顶点可着色的最小数值 k。若 $\chi(G)=k$，则称 G 是 k-色图。图 G 的每一个 k-顶点着色 f 对应于顶点集 V 的一个划分 $\{V_1,V_2,\cdots,V_k\}$，也记作 $f=(V_1,V_2,\cdots,V_k)$，V_i 为色组，表示分配到颜色 i 的所有顶点构成的集合，即

$$V(G)=\bigcup_{i=1}^{k}V_i,\ V_i\neq\phi,\ V_i\cap V_j=\phi,\ i\neq j,\ i,j=1,2,\cdots,k$$

其中，$V_i\ (i=1,2,\cdots,k)$ 是 G 的独立集。图 G 中所有不同的 k-着色所构成的集合用 $C_k(G)$ 表示。用 $C_k^0(G)$ 表示 G 的所有由 k 个色组构成的划分的集，简称为图 G 的 k-**色组划分集**。

如果一个图能够画在平面上使得它的边仅在顶点相交，则称这个图为**平面图**。平面图的这样一种画法称为它的一个**平面嵌入**，本文所研究的平面图均是指基于它的一个平面嵌入下的平面图。对于一个平面图，如果只要任何两个不相邻的顶点之间再加一条边，其平面性一定被破坏，则称该平面图为**极大平面图**。若一个平面图的每个面(包括无穷面)都由 3 条边界组成，则称该平面图为**三角剖分图**。易证，极大平面图和三角剖分图是等价的。

无论是理论上还是应用上，平面图都是一类非常重要的图类。理论上，著名的 4-色猜想，唯一 4-色平面图猜想，9-色猜想，以及 3-色问题等[1]不仅在图论领域，乃至整个数学界都具有重大影响；从应用的角度来讲，平面图理论可直接应用于电路布线[2]，信息科学[3]等领域。

由于著名的 4-色猜想的研究对象可归为极大平面图，因此，100 多年来，关于极大平面图的研究吸引了众多的学者，围绕着 4-色猜想，相继研究了诸如度序列、构造、着色特性、可遍历性，生成运算等诸多方面[4]。并且在研究过程中，提出了诸如唯一 4-色极大平面图猜想以及 9-色猜想等，它们也逐渐构成了极大平面图着色理论的核心研究目标。

在 4-色猜想的研究过程中，从 1879 年 KEMPE 的“证明”[5]，到 HEAWOOD 的反例[6]，再到 1976 年由 HAKEN 与 APPEL 给出的“计算机证明”[7–9]，主要集中在“寻找可约的不可避免集”这一种研究方法上。利用这种方法通过电子计算机找到了 1936 个可约的不可避免集，证明了四色猜想成立；1997 年由 ROBERTSON, SANDERS, SEYMOUR 和 THOMAS 等人找出了 633 个可约构形的不可避免集，简化了四色猜想的计算机证明[10,11]。

“不可避免集”的研究起源于 1904 年 WERNICKE 的工作[12];“可约构形”是 BIRKHOFF 于 1913 年提出来的[13]。在利用“寻找可约的不可避免集”这种思想的证明过程中，德国数学家 HEESCH 做出了不可磨灭的贡献，他发现了证明可约性的一种方法——**放电法**[14]，并确信此方法可解决四色猜想，为 1976 年利用电子计算机求解四色猜想奠定了基础。另外，还有不少学者在此方法上作出贡献，诸如 FRANKLIN[15,16], REYNOLDS[17], WINN[18], ORE 和 STEMPLE[19], MAYER[20]。

人是无法通过手工对不可避免集和可约构形进行验证的，因此，如何给出数学证明**仍是一个尚待解决的数学难题**。

除了基于“可约的不可避免集”的证明方法外，另一个具有影响的证明方法是基于假设：“每个 3-正则 3-连通的平面图都有哈密顿圈”的条件给出的“证明”，该证明是由 TAIT 于 1880 年给出[21]。由于这个假设是错的，其证明自然是错的。这个错误的假设是 PETERSEN[22]发现的，但直到 1946 年，TUTTE 才找到该证明的反例[23]。后来，GRINBERG[24]在 1968 年找到了一个必要条件，由此可产生很多 3-正则 3-连通的非 HAMILTON 平面图。虽然 TAIT 的证明是错的，但 TAIT 的工作对于图论的研究，特别是边着色理论产生了深远的影响。用 $f(G,t)$ 表示对标定图 G 的顶点用 t 种颜色进行着色时具有的着色数目。显然，当 $t<\chi(G)$ 时，即该图没法被着色时，$f(G,t)=0$；但当 $\chi(G)\le t$，这种着色的数目肯定存在，即 $f(G,t)>0$。对于任意一个平面图 G，当 $t=4$ 时，若能够证明 $f(G,4)>0$，则就相当于证明了四色猜想！这就是

BIRKHOFF 在 1912 年提出用来证明四色猜想的一种方法[25,26]。后来发现，$f(G,t)$ 是一个关于 t 的多项式，故称 $f(G,t)$ 为图的**色多项式**。目前，图的色多项式已经成为了图论学科中一个很有魅力的独立分支[27]。遗憾的是，BIRKHOFF 的愿望至今尚未实现。对色多项式的研究引起了众多学者的极大兴趣。关于这方面研究较为深入的文章与专著可参阅文献[25−31]，其中文献[28]的结果最为诱人，证明了：当 $t=\tau(\sqrt{5})=3.618\cdots$ (其中 $\tau=(\sqrt{5}+1)/2$)时，$f(G,\tau\sqrt{5})>0$。此结果与四色猜想有点“擦肩而过”的遗憾，因为只要能够证明 $f(G,4)>0$，则四色猜想成立。

在计算给定图的色多项式方面，一个最为基本的公式是所谓的**缩边递推公式**。

对于图 G 中的一条边 e，用 $G-e$ 和 $G\circ e$ 分别表示图 G 经过对边 e 进行删边运算和收缩运算后得到的图。在收缩运算中，假设除 W_2 以外，图 G 是无自环且没有平行边的。

缩边递推公式[25] 若图 G 是简单图，则对图 G 的任何边 e，都有

$$f(G,t)=f(G-e,t)-f(G\circ e,t)$$

另外，本文作者在文献[32,33]分别给出缩点递推公式以及图与补图的色多项式等。

可能是因为 TUTTE 的工作很漂亮，以及 TUTTE 在学术界的地位，人们认为以图的色多项式为工具解决四色猜想似乎不可能，本文下述的工作重新“燃起”了利用图的色多项式作为工具之一来证明四色猜想的希望。

2　色多项式的缩轮递推公式

为方便，先给出如下 2 个引理：

引理 1[26] 对任何无自环的平面图 G，G 是 4-可着色的当且仅当

$$f(G,4)>0 \tag{1}$$

引理 2[25,27] 若图 G 是子图 G_1 与 G_2 的并，且 G_1 与 G_2 的交为 k-阶完全图，则

$$f(G,t)=\frac{f(G_1,t)\times f(G_2,t)}{t(t-1)\cdots(t-k+1)} \tag{2}$$

定理 1 设图 G 是一个极大平面图。v 是图 G 的一个 4 度顶点，且 $N(v)=\{v_1,v_2,v_3,v_4\}$，如图 2 所示，则有

$$f(G,4)=f(G_1,4)+f(G_2,4) \tag{3}$$

其中 $G_1=(G-v)\circ\{v_1,v_3\}$，$G_2=(G-v)\circ\{v_2,v_4\}$。

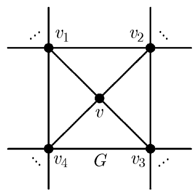
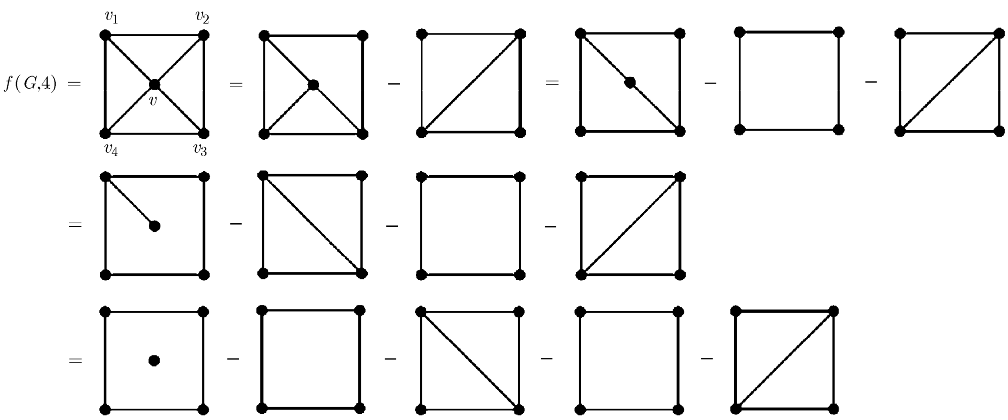


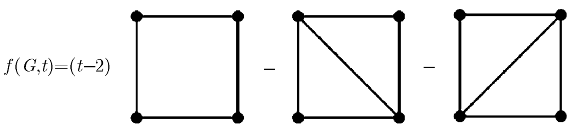
图 2 一个含有 4 度顶点的极大平面图

证明 在下面的推导过程中，用记号 $G[\overline{N(v)}]$ 来代表图 G。现在我们应用缩边递推公式来求图 G 的色多项式。为了直观，我们采用 ZYKOV[34]引入的一种方法，先不写 t，而用图的一个图解来记它的色多项式。注意：若有一对顶点间至少有 2 条边，除 W_2 外，只保留一条，删去其余的边。



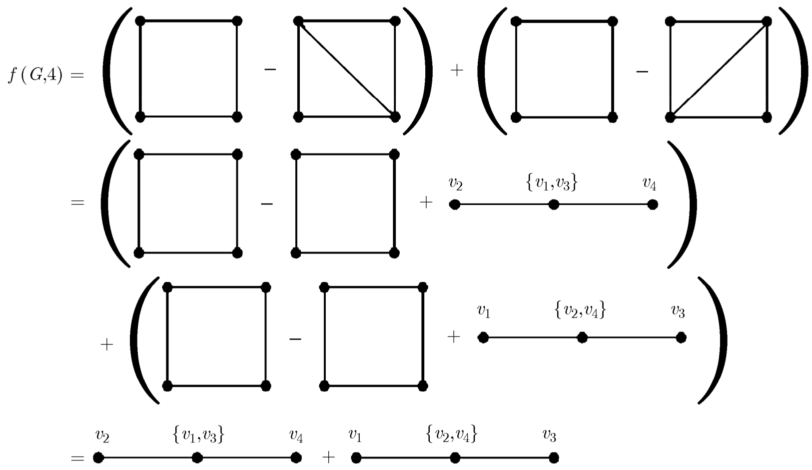
(4)

由引理 2，式(4)中第 1 个图的色多项式应该为 $t\cdot f(G-v,t)$，因此，我们有



(5)

当 $t=4$ 时，有



(6)

注意到式(6)等号右边的两个图实际上分别表示 $(G-v)\circ\{v_1,v_3\}$ 和 $(G-v)\circ\{v_2,v_4\}$。很容易证明，它们都是顶点数为 $n-2$ 的极大平面图。因此有

$$f(G,4)=f\big((G-v)\circ\{v_1,v_3\},4\big)+f((G-v)\circ\{v_2,v_4\},4)=f(G_1,4)+f(G_2,4) \tag{7}$$

即

$$f(G,4)=f(G_1,4)+f(G_2,4) \tag{8}$$

从而本定理获证。

定理 2　设图 G 是一个极大平面图。v 是图 G 的一个 5 度顶点，且 $N(v)=\{v_1,v_2,v_3,v_4,v_5\}$，其结构如图 3 所示。则有

$$\begin{aligned}f(G,4)=&[f(G_1,4)-f(G_1\cup\{v_1v_4,v_1v_3\},4)]\\&+[f(G_2,4)-f(G_2\cup\{v_3v_1,v_3v_5\},4)]\\&+[f(G_3,4)-f(G_3\cup\{v_1v_4\},4)]\end{aligned} \tag{9}$$

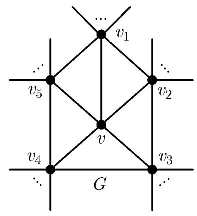
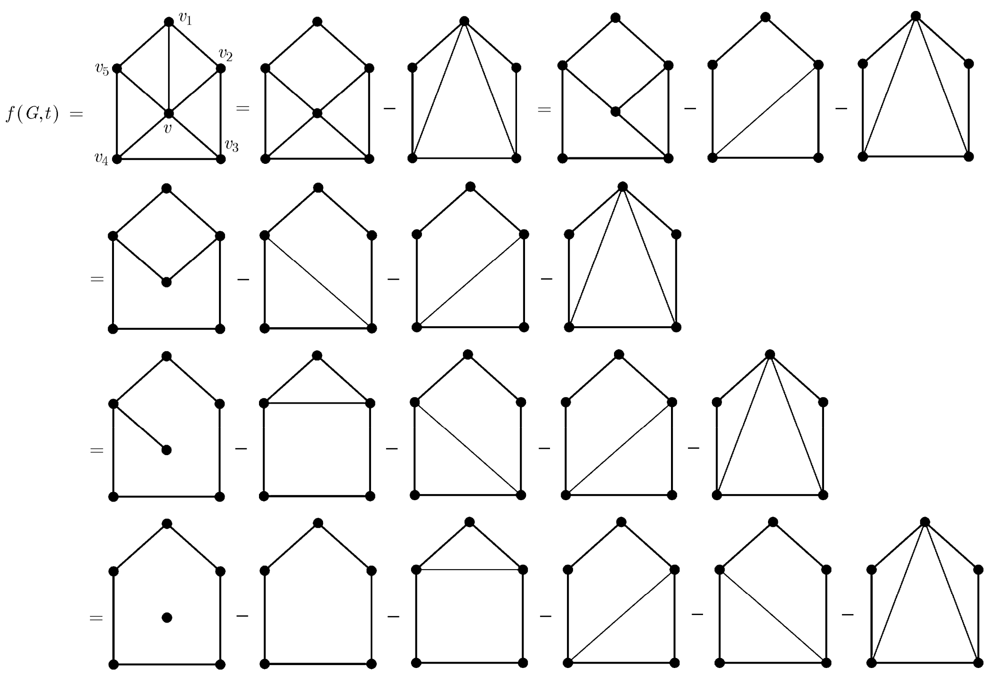


图 3　一个含有 5 度顶点的极大平面图

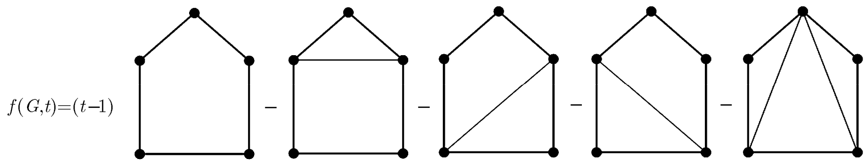
其中 $G_1=(G-v)\circ\{v_2,v_5\}$，$G_2=(G-v)\circ\{v_2,v_4\}$，$G_3=(G-v)\circ\{v_3,v_5\}$。

证明　利用 $G[\overline{N(v)}]=G[v_1,v_2,v_3,v_4,v_5,v]$ 来代表图 G。现对图 G 反复应用缩边递推公式，在运算过程中若产生多重边，则删去重边只保留一条边，但轮图 W_2 除外，且用轮图 W_5 来表示极大平面图 G 的色多项式。



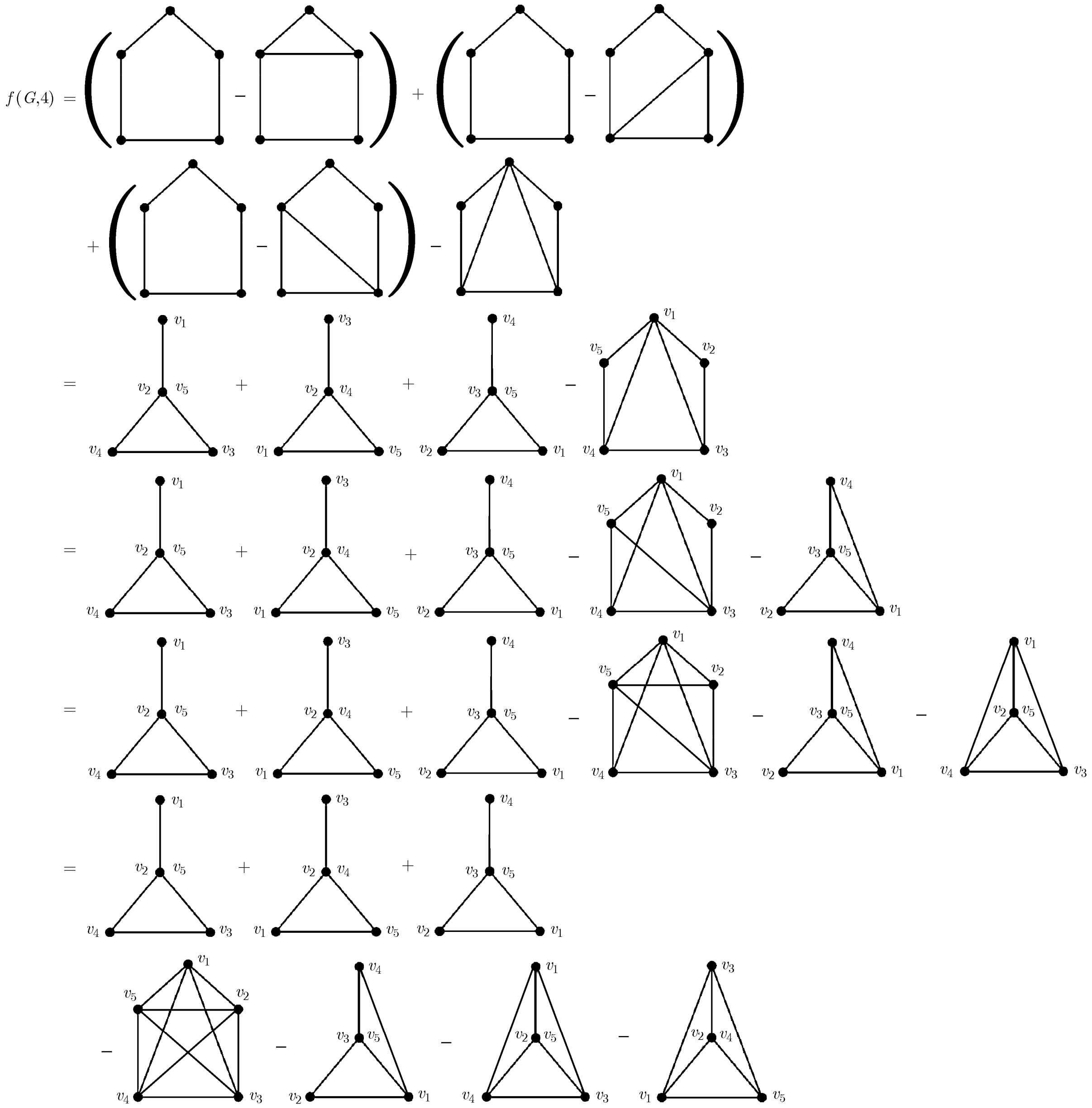
(10)

由引理 2，式(10)最后一个等号右端第 1 个图的色多项式应该为 $t\cdot f(G-v,t)$。因此，我们有



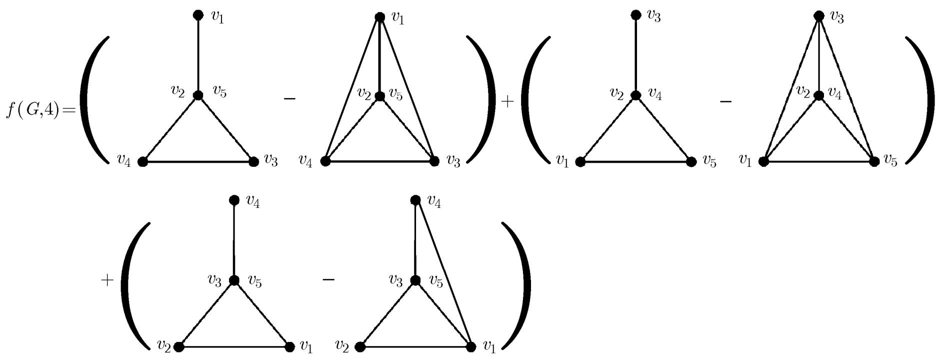
(11)

取 $t=4$ 时，我们有



(12)

注意到在式(12)最后一个等号右端的第 4 个图，记作图 G'，它含有子图 K_5，故 $f(G',4)=0$。由此，我们可得到式(13)



(13)

注意到式(13)等号右端第 1 个括号内的第 1 个图实际上就是图$G_1=(G-v)\circ\{v_2,v_5\}$；第 2 个括号内的第 1 个图实际上是$G_2=(G-v)\circ\{v_2,v_4\}$；第 3 个括号内的第 1 个子图实际上是$G_3=(G-v)\circ\{v_3,v_5\}$，从而本定理获证。

3 定理 2 给出了证明四色猜想的一种思路

众所周知，四色猜想的最终证明一般需采用数学归纳法，且按照最小度进行分类。当最小度为$\delta(G)=3,4$时，由归纳法容易证明，但当$\delta(G)=5$时，至今数学方法尚未给出证明。下面，给出一种**基于定理 1 和定理 2 的四色猜想证明思路：**

欲证对任一极大平面图G，$f(G,4)>0$，现对其顶点数n施行归纳法。

当$n=3,4,5$时，显然结论成立。

假设当顶点数≥ 5而$\leq n-1$时结论成立。我们来考察顶点数为n的情况。由于我们只考虑简单极大平面图。而任一极大平面图G满足：$3\leq\delta(G)\leq 5$。故按最小度分下列 3 种情况考虑。

情况 1 $\delta(G)=3$：

设$v\in V(G),d(v)=3$。令$G_1=G[\overline{N(v)}]$，$G_2=G-v$，故$G_1\bigcap G_2=G[N(v)]\cong K_3$；

再注意到$G_1=G[\overline{N(v)}]\cong K_4$，于是由引理 2 有

$$f(G,t)=f(G_1\bigcup G_2,t)=\frac{f(G_1,t)\times f(G_2,t)}{f(K_3,t)}=(t-3)f(G_2,t)$$

由归纳假设，$f(G_2,4)>0$，故有：$f(G,4)=f(G_2,4)>0$。

这就证明了$\delta(G)=3$时结论成立。

情况 2 $\delta(G)=4$：

设$v\in V(G),d(v)=4$。$N(v)=\{v_1,v_2,v_3,v_4\}$，其结构如图 2 所示。注意，在图中用$G[\overline{N(v)}]$来代表图G。由定理 1 知，$f(G,4)=f(G_1,4)+f(G_2,4)$，其中$G_1=(G-v)\circ\{v_1,v_3\}$，$G_2=(G-v)\circ\{v_2,v_4\}$。易证，$G_1$和$G_2$都是顶点数为$n-2$的极大平面图，故由归纳假设有

$$f(G_1,4)=f\left((G-v)\circ\{v_1,v_3\},4\right)>0$$

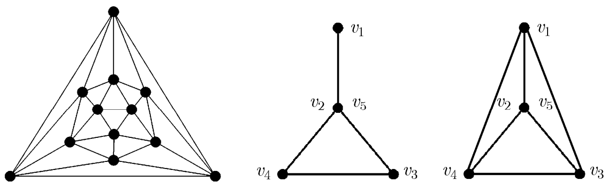
$$f(G_2,4)=f\left((G-v)\circ\{v_2,v_4\},4\right)>0$$

从而有：$f(G,4)=f(G_1,4)+f(G_2,4)>0$，故$\delta(G)=4$时结论成立。

关键是下面的情况 3。

情况 3 $\delta(G)=5$：

顶点数最少的最小度为 5 的极大平面图是正 20 面体，如图 4(a)所示，具有 12 个顶点，它显然是 4-可着色的。不存在 13 阶的最小度为 5 的极大平面图。故对顶点数≥ 14，且最小度为 5 的极大平面图



(a)正20面体 (b)漏斗子图 (c) K_4

图 4 情况 3 说明示意图

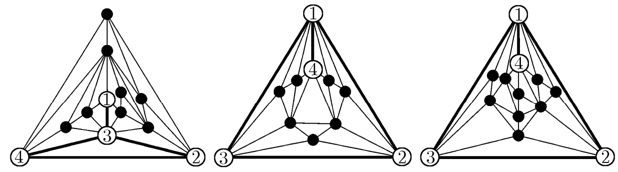
G，一定$\exists v\in V(G)$，$d(v)=5$，其邻域$N(v)=\{v_1,v_2,v_3,v_4,v_5\}$中的顶点$v_1$(结构如图 3 所示)满足$d_G(v_1)\geq 6$。于是，定理 2 中的图$G_1$(如图 4(b)示)，它是一个最小度$\geq 4$的 4-色极大平面图。

基于此约定，欲证$f(G,4)>0$，有两种如下思路：

第 1 种思路：显然，由于

$$\begin{aligned}f(G,4)=&[f(G_1,4)-f(G_1\cup\{v_1v_4,v_1v_3\},4)]\\&+[f(G_2,4)-f(G_2\cup\{v_3v_1,v_3v_5\},4)]\\&+[f(G_3,4)-f(G_3\cup\{v_1v_4\},4)]\end{aligned}$$

中每个括号内的值≥ 0。若其中一个>0，则四色猜想成立。而第 1 个括号内的值$>0\Leftrightarrow\exists f_1\in C_4^0(G_1)$，$f_1(v_1)=f_1(v_3)$或$f_1(v_1)=f_1(v_4)$。故使式(9)的值为$0\Leftrightarrow$每个括号中的值为 0。而使第 1 个括号内的值为 0 的充要条件是：$\forall f_1\in C_4^0(G_1)$，$f_1(v_1)\neq f_1(v_3)$，且$f_1(v_1)\neq f_1(v_4)$，即对$\forall f_1\in C_4^0(G_1)$，如图 4(b)所示的**漏斗子图**中每个顶点的着色两两互不相同。我们把这类图称为**4-色漏斗型-伪唯一 4-色极大平面图**，如图 5 中所示的 3 个图，均为 4-色漏斗型伪唯一 4-色极大平面图。



(a) 12阶4-色漏斗型 (b) 11阶4-色漏斗型 (c) 13阶4-色漏斗型

图 5 3 个 4-色漏斗型-伪唯一 4-色极大平面图

一个k-色图G称为**可k-色坐标系的**，如果在G中存在k个顶点$v_1,v_2,\cdots,v_k$，使得对G的任意k-着色f，$f(v_1),f(v_2),\cdots,f(v_k)$两两互不相同。对于可 4-色坐标系的极大平面图图，共分为 3 类：(1)唯一 4-色极大平面图，即只有一种色组划分的极大平面图；(2)拟唯一 4-色极大平面图，即含有唯一 4-色极大平面图的子图；(3)伪唯一 4-色极大平面图，即非唯一 4-色极大平面图，又非拟唯一 4-色极大平面图的 4-色极大平面图，此方面更为深入的研究，将在本系列文章的后续文章中给出。

第 2 种思路：定理 2 中给出的 3 个极大平面图 G_1, G_2, G_3 可视为：在原图 G 中，首先删去顶点 v，即为 $G-v$，进而在此基础上分别收缩 5-圈上的顶点对 $\{v_2,v_5\}, \{v_2,v_4\}, \{v_3,v_5\}$ 得到的极大平面图，如图 6 示，且 G 中的 5-圈分别收缩成 G_1, G_2, G_3 中的漏斗子图，分别记作 $L_1=v_1-\Delta v_2^5 v_3 v_4$, $L_2=v_3-\Delta v_2^4 v_1 v_5$, $L_3=v_4-\Delta v_3^5 v_1 v_2$，如图 6 中下方 3 个图示，$v_2^5, v_2^4, v_3^5$ 分别代表收缩点对 $\{v_2,v_5\}, \{v_2,v_4\}$，$\{v_3,v_5\}$ 后得到的新顶点。

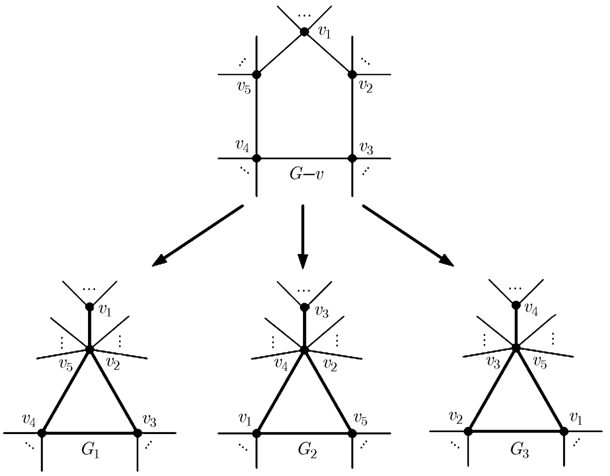


图 6　3 个漏斗子图的产生过程说明示意图

由归纳假设，G_1，G_2，G_3 均为 4-可着色的。欲证 $f(G,4)>0$，只要证这 3 个极大平面图中 3 个漏斗子图 L_1，L_2，L_3 中至少有一个不是 4-色漏斗子图即可。

由此给出**证明四色猜想的第 2 种思路**：对于一个最小度为 5 的极大平面图 G 中的 5-轮 W_5^v，对应于 G_1，G_2，G_3 的 3 个漏斗子图 L_1，L_2，L_3 至少有一个为非 4-色漏斗。例如，我们视图 5(a)是由图 7 中的 5-轮 $W_5^v=v-v_1v_2v_3v_4v_5$ 按图 6 所示的方法获得的，容易证明，按图 6 中所示的方法获得的其余两个极大平面图不含 4-色漏斗。

4　结束语

本文给出了求解极大平面图的一种色多项式的递推公式，由该公式发现：第一，证明四色猜想的两种思路；第二，导出了一种将图的着色与构造相融合的构造极大平面图的方法——**扩缩运算系统**，如图 5(a)就是通过所谓的扩 5-轮运算获得图 7 中所示的极大平面图，或者说，图 7 中所示的极大平面图是通过缩 5-轮运算得到图 5(a)。极大平面图的扩缩运算系统的详细研究将在本系列文章的第 2 篇中给出。

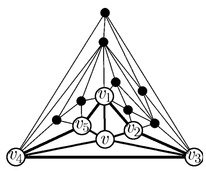


图 7　可收缩成图 5 中第 1 个图的极大平面图

致谢：本文在完成过程中相继与我的 5 位图论专业学生：朱恩强与李泽鹏博士后；刘小青与王宏宇博士生以及周洋洋硕士生等进行了多次有益的讨论，在此表示感谢。

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Theory on Structure and Coloring of Maximal Planar Graphs

(1)Recursion Formulae of Chromatic Polynomial and Four-Color Conjecture

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Abstract: In this paper, two recursion formulae of chromatic polynomial of a maximal planar graph G are obtained: when $\delta(G)=4$, let W_4^v be a 4-wheel of G with wheel-center v and wheel-cycle $v_1v_2v_3v_4v_1$, then $f(G,4)=f((G-v)\circ\{v_1,v_3\},4)+f((G-v)\circ\{v_2,v_4\},4)$; when $\delta(G)=5$, let W_5^v a 5-wheel of G with wheel-center v and wheel-cycle $v_1v_2v_3v_4v_5v_1$, then $f(G,4)=[f(G_1,4)-f(G_1\cup\{v_1v_4,v_1v_3\},4)]+[f(G_2,4)-f(G_2\cup\{v_3v_1,v_3v_5\},4)]+[f(G_3,4)-f(G_3\cup\{v_1v_4\},4)]$, $G_1=(G-v)\circ\{v_2,v_5\}$, $G_2=(G-v)\circ\{v_2,v_4\}$, $G_3=(G-v)\circ\{v_3,v_5\}$, where"$\circ$"denotes the operation of vertex contraction. Moreover, the application of the above formulae to the proof of Four-Color Conjecture is investigated. By using these formulae, the proof of Four-Color Conjecture boils down to the study on a special class of graphs, viz., 4-chromatic-funnel pseudo uniquely-4-colorable maximal planar graphs.

Key words: Four-Color Conjecture; Maximal planar graphs; Chromatic polynomial; Pseudo uniquely-4-colorable planar graphs; 4-chromatic-funnel

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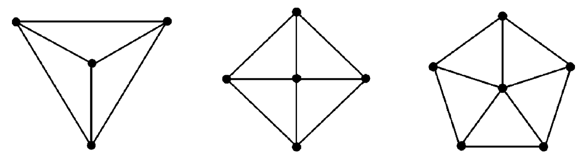
1 Introduction

All graphs considered in this paper are finite, simple, and undirected. For a given graph G, we use $V(G)$, $E(G)$, $d_G(v)$, and $N_G(v)$ to denote the vertex set, the edge set, the degree of v, and the neighborhood of v in G (the set of neighbors of v), respectively, which can be written as V, E, $d(v)$, and $N(v)$ for short. The order of G is the number of its vertices. A graph $H=(V',E')$ is a subgraph of G if $V'\subseteq V$ and $E'\subseteq E$. For a subgraph H of G, whenever $u,v\in V'$ are adjacent in G, they are also adjacent in H, then H is an induced subgraph of G or a subgraph of G induced by V', denoted by $G[V']$. Two graphs G and H are disjoint if they have no vertex in common. By starting with a disjoint union of G and H, and adding edges joining every vertex of G to every vertex of H, one obtains the join of G and H, denoted by $G\vee H$. We write K_n and C_n for the complete graph and cycle of order n, respectively. The join $C_n\vee K_1$ of a cycle and a single vertex is referred to as a wheel, denoted by W_n (the examples W_3,W_4,W_5 are shown in Fig. 1, where C_n is called the cycle of W_n and the vertex of K_1 is called the center of W_n. If $V(K_1)=\{x\}$, we also denote the cycle of W_n by C^x. A graph is k-regular if all of its vertices have the same degree k. A 3-regular graph is usually called a cubic graph.



(a) W_3 (b) W_4 (c) W_5

Fig. 1 Three wheels W_3,W_4,W_5

A k-coloring of a graph G is a mapping f from the vertex set V to the color set $C(k)=\{1,2,\cdots,k\}$ such that $f(x)\neq f(y)$ for any $xy\in E(G)$. A graph G is k-colorable if it admits a k-coloring. The minimum k for which a graph G is k-colorable is called its chromatic number, denoted by $\chi(G)$. If $\chi(G)=k$, G is called a k-chromatic graph. Alternatively, each k-coloring f

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of G can be viewed as a partition $\{V_1, V_2, \cdots, V_k\}$ of V, where V_i denotes the set of vertices assigned color i, called a color class of f. So it can be written as $f=(V_1, V_2, \cdots, V_k)$. In other words,

$$V(G)=\bigcup_{i=1}^{k} V_i,\ V_i \neq \varnothing,\ V_i \cap V_j = \varnothing,\ i \neq j,\ i,j=1,2,\cdots,k$$

where V_i is an independent set of G, $i=1,2,\cdots,k$. The set of all k-colorings of a graph G can be denoted by $C_k(G)$. For a k-chromatic graph G, the notation $C_k^0(G)$ denotes the set of all the partitions of k-coloring class of G, simplified by the partition set of k-color class of G.

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a planar embedding of the graph. Any planar graph considered in the paper is under its planar embedding. A maximal planar graph is a planar graph to which no new edges can be added without violating planarity. A triangulation is a planar graph in which every face is bounded by three edges (including its infinite face). It can be easily proved that maximal planar graphs are triangulations, and vice versa.

The planar graph is a very important class of graphs no matter which aspect, theoretical or practical, is concerned. In theory, there are many famous conjectures that have very significant effect on graph theory, even mathematics, such as the Four-Color Conjecture, the Uniquely Four-Colorable Planar Graphs Conjecture, the Nine-Color Conjecture, Three-Color Problem, *etc*[1]. In application, planar graphs can be directly applied to the study of layout problems[2], information science[3], *etc*.

Because the studying object of the well-known Four-Color Conjecture can be confined to maximal planar graphs, many scholars have been strongly attracted to the study of this typical topic. They did research on maximal planar graphs from a number of different standpoints, such as degree sequence, construction, coloring, traversability, generating operations, *etc*[4]. Moreover, many new conjectures on maximal planar graphs have been proposed, for instance, Uniquely Four-Colorable Planar Graphs Conjecture and Nine-Color Conjecture. These conjectures have gradually become the essential topics on maximal planar graphs.

In the process of studying Four-Color Conjecture, one important method, finding an unavoidable set of reducible configurations, was proposed. This method has been used in Kempe's "proof"[5], Heawood's counterexample[6], and the computer-assisted proof due to Appel and Haken[7−9]. Using this method, Appel and Haken found an unavoidable set containing 1936 reducible configurations and proved Four-Color Conjecture. In 1997, Robertson, *et al.*[10,11] gave a simplified proof. They found an unavoidable set containing only 633 reducible configurations.

The research on unavoidable sets originated from Wernicke's work[12] in 1904. The concept of reducibility was introduced by Birkhoff[13] in 1913. On the research for finding an unavoidable set of reducible configurations, the great contribution was made by German mathematician Heesch[14]. He introduced a method "discharging" to find an unavoidable set of a maximal planar graph, which lied the foundation for solving Four-Color Conjecture by electronic computer in 1976[7−9]. Moreover, many researchers studied Four-Color Problem by this method, such as Franklin[15,16], Reynolds[17], Winn[18], Ore and Stemple[19], and Mayer[20].

However, these proofs were all computer-assisted and hard to be checked one by one by hand. Therefore, finding a mathematical method to concisely solve the Four-Color Problem is still an open hard problem.

Another incorrect proof of Four-Color problem[21] was given by Tait in 1880. His proof was based on an assumption: each 3-connected cubic plane graph was Hamiltonian. Because this assumption is incorrect, Tait's proof is incorrect.

Although the error in his proof was found by Petersen[22] in 1898, the counterexample was not given until 1946[23]. Then, in 1968, Grinberg[24] obtained a necessary condition, thus producing many non-Hamiltonian cubic planar graphs of 3-connected. Although the proof of Tait was incorrect, his work had a strong influence on the research on Graph Theory, especially edge-coloring theory.

Let $f(G,t)$ be the number of colorings for the vertices of a labeled graph G with t colors. Obviously, if $t<\chi(G)$, G can not be properly colored, so $f(G,t)=0$. But if $\chi(G)\le t$, then G admists this coloring must exist, and that is $f(G,t)>0$. For every planar graph G, if $f(G,4)>0$ can be proved, it is equivalent to the proof of the Four-Color Problem! This is the method that Birkhoff[25,26] had proposed for attacking Four-Color Problem in 1912. Later on, it was found that $f(G,t)$ is a polynomial in the number t, called the chromatic polynomial of graphs, which has become a fascinating branch in the field of graph theory at present[27]. But it was a pity that Birkhoff's aim had not been reached. Further research on chromatic polynomials can be found in Refs. [25–31]. The best result, due to Tutte[28], was that if $t=\tau(\sqrt{5})=3.618\cdots$ (where $\tau=(\sqrt{5}+1)/2$), then $f(G,\tau\sqrt{5})>0$. The result seemed to be a pity that it brushed past the Four-Color Problem, because the Four-Color Conjecture holds if $f(G,4)>0$.

In order to calculate the chromatic polynomial of a given graph, the basic tool is the Deletion-Contract Edge Formula.

For an edge e of a graph G, we denote by $G-e$ and $G\circ e$ the graphs obtained from G by deleting and contracting the edge e, respectively. Throughout the contraction operation, all graphs have no loops and parallel edges, except W_2.

The Deletion-contract Edge Formula. For a given graph G and an edge $e\in E(G)$, we have

$$f(G,t)=f(G-e,t)-f(G\circ e,t)$$

Moreover, XU etal. [32,33] obtained a recursion formula of chromatic polynomial by vertex deletion and a chromatic polynomial between a graph and its complement.

Perhaps for the perfect degree of Tutte's work and his highly status in academia, once upon a time, it was thought that to attack the Four-Color Problem by chromatic polynomial is impossible. Nevertheless, our work below gives a new hope to solve the Four-Color Problem by chromatic polynomial.

2 Recursion Formulae of Chromatic Polynomial by Contracting Wheels

We first give two useful lemmas as follows.

Lemma 1[26] For any planar graph G, it is 4-colorable if and only if

$$f(G,4)>0 \tag{1}$$

Lemma 2[25,27] Let G be the union of two subgraphs G_1 and G_2, whose intersection is a complete graph of order k. Then

$$f(G,t)=\frac{f(G_1,t)\times f(G_2,t)}{t(t-1)\cdots(t-k+1)} \tag{2}$$

Theorem 1 Let G be a maximal planar graph, v be a 4-degree vertex of G, and $N(v)=\{v_1,v_2,v_3,v_4\}$ (see Fig. 2). Then

$$f(G,4)=f(G_1,4)+f(G_2,4) \tag{3}$$

where $G_1=(G-v)\circ\{v_1,v_3\}$, $G_2=(G-v)\circ\{v_2,v_4\}$。

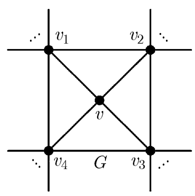
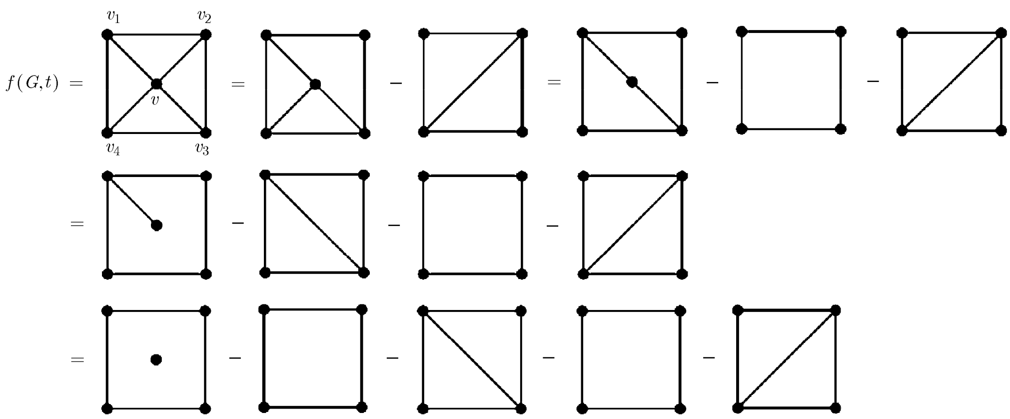


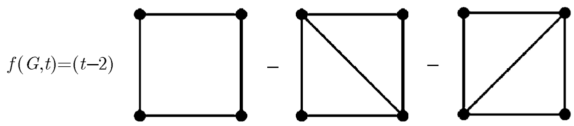
Fig. 2 A maximal planar graph with a 4-degree vertex

Proof In the following derivation, we represent G by $G[\overline{N(v)}]$. Now we first compute the chromatic polynomial of the graph G by the Deletion-Contract Edge Formula. For the sake of understanding clearly, a method introduced by Zykov[34] is used here, where the chromatic polynomials are represented by the corresponding graphical graphs without t. Notice that if there are at least two edges adjacent to two vertices, then only one remains and others are deleted excluding W_2.



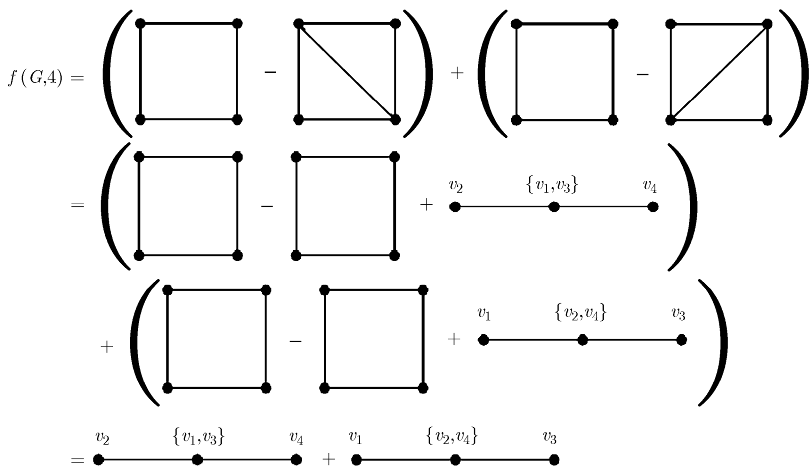
(4)

By Lemma 2, the chromatic polynomial of the first subgraph in Formula (4) is $t\cdot f(G-v,t)$. Therefore,



(5)

When $t=4$, we can obtain that



(6)

Notice that the two graphs in Formula (6) denote $(G-v)\circ\{v_1,v_3\}$ and $(G-v)\circ\{v_2,v_4\}$, respectively. It is easily proved that they are both maximal planar graphs of order $n-2$. Thus, we obtain that

$$f(G,4)=f\big((G-v)\circ\{v_1,v_3\},4\big)+f((G-v)\circ\{v_2,v_4\},4)=f(G_1,4)+f(G_2,4) \tag{7}$$

namely

$$f(G,4)=f(G_1,4)+f(G_2,4) \tag{8}$$

Theorem 2 Let G be a maximal planar graph, v be a 5-degree vertex of G, and $N(v)=\{v_1,v_2,v_3,v_4,v_5\}$ (see Fig. 3). Then

$$\begin{aligned}f(G,4)=&[f(G_1,4)-f(G_1\cup\{v_1v_4,v_1v_3\},4)]\\&+[f(G_2,4)-f(G_2\cup\{v_3v_1,v_3v_5\},4)]\\&+[f(G_3,4)-f(G_3\cup\{v_1v_4\},4)]\end{aligned} \tag{9}$$

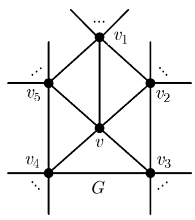
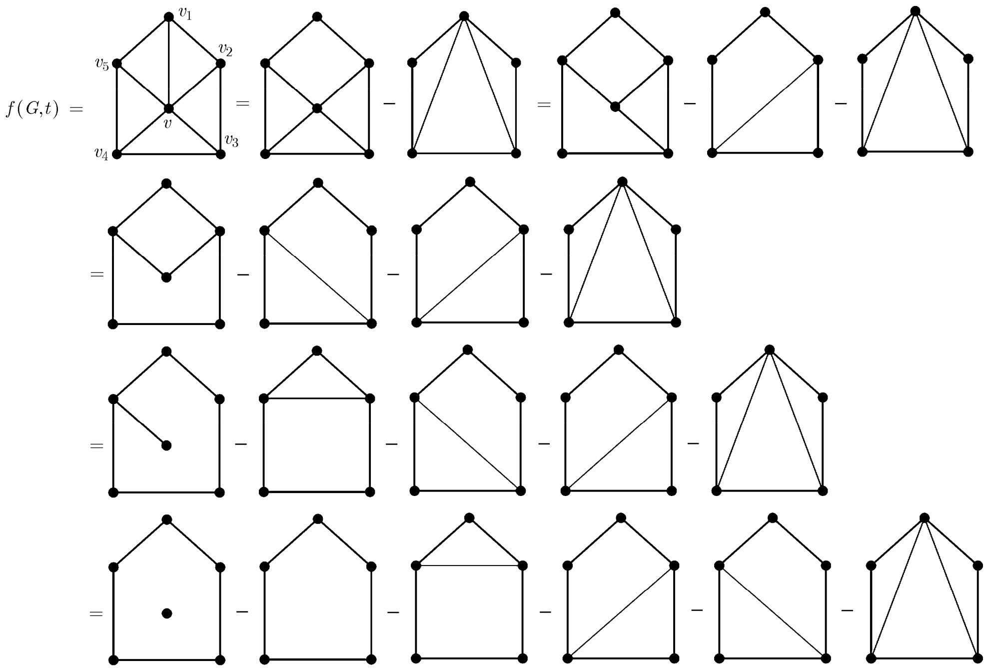


Fig. 3 A maximal planar graph with a 5-degree vertex

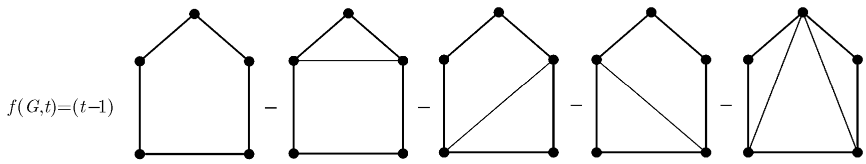
where $G_1=(G-v)\circ\{v_2,v_5\},G_2=(G-v)\circ\{v_2,v_4\}$, $G_3=(G-v)\circ\{v_3,v_5\}$ 。

Proof The graph G is represented by $G[\overline{N(v)}]$ in the following proof. The chromatic polynomial of graph G can be calculated by applying the Deletion-Contract Edge Formula repeatedly. If parallel edges appear in the process, reserve only one edge excluding W_2 . We use W_5 to represent the chromatic polynomial of G . In this way, we can obtain that



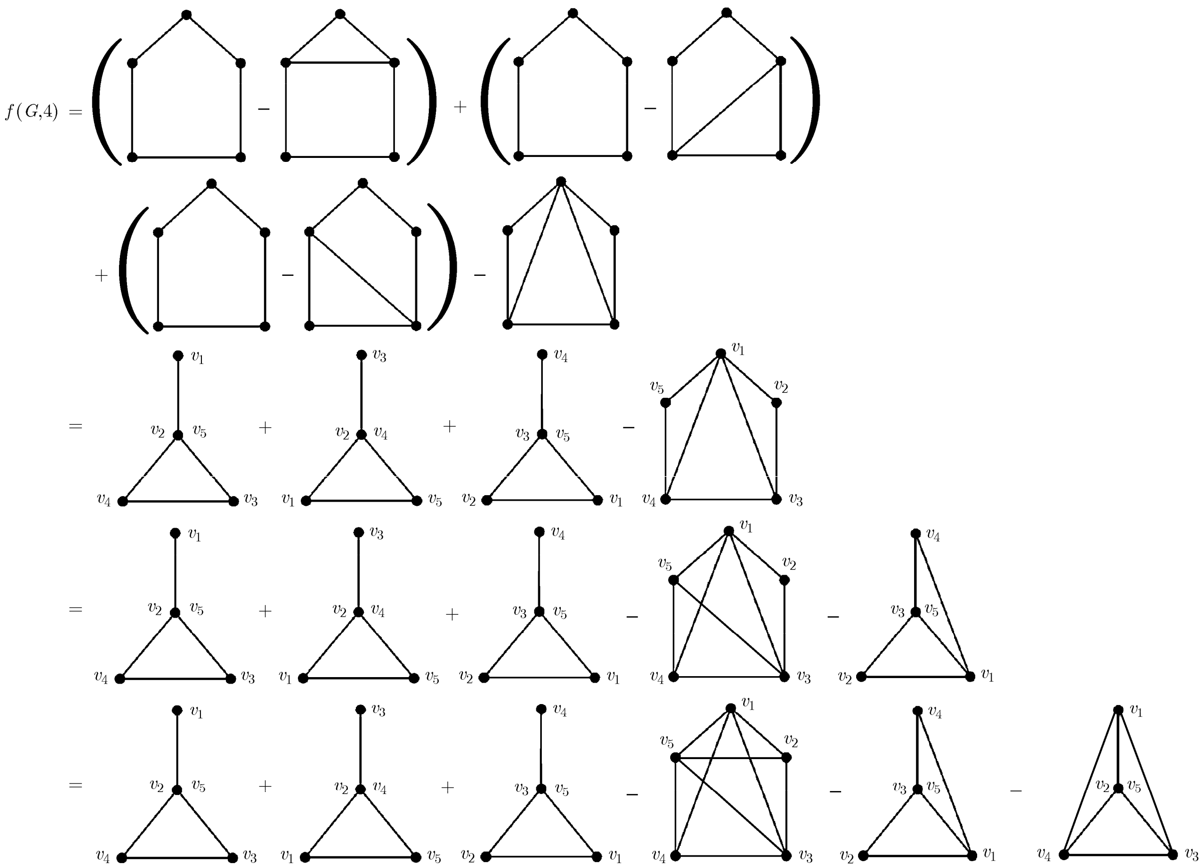
(10)

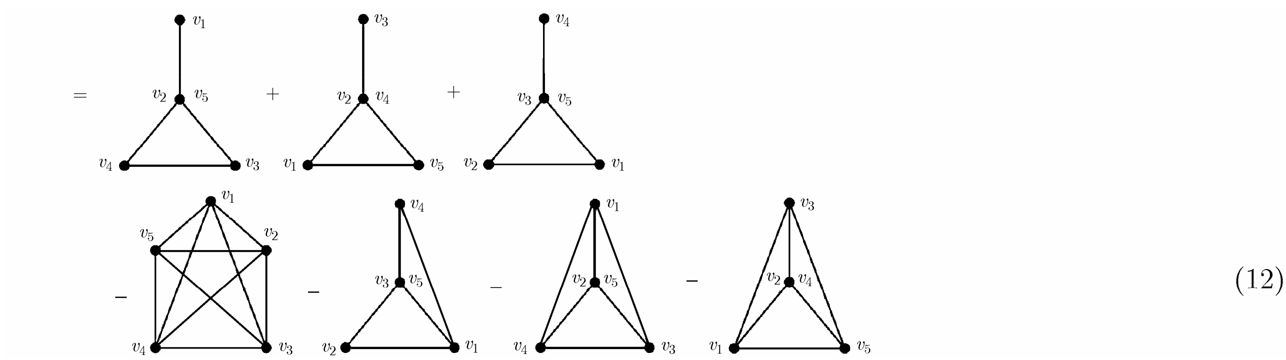
By Lemma 2, the chromatic polynomial of the first graph in Formula (10) is $t \cdot f(G-v,t)$. Therefore, we can obtain that



(11)

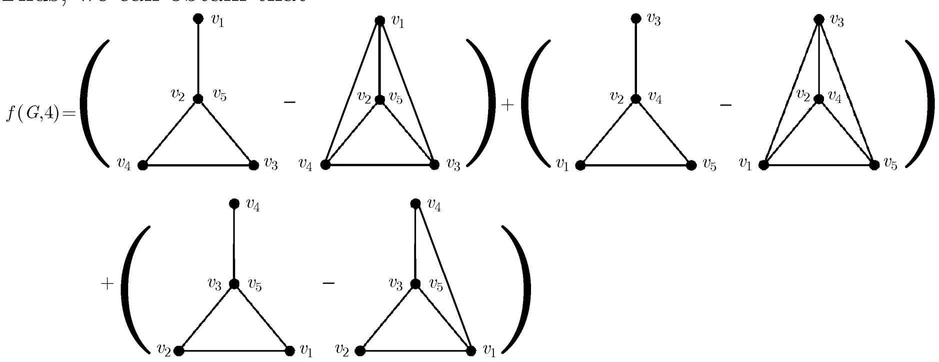
When $t=4$, we have





(12)

Notice that the fourth graph in Formula (12), denoted by G', contains a subgraph K_5, and so $f(G',4)=0$. Thus, we can obtain that



(13)

Actually, the first graph in the first bracket of Formula (13) is $G_1=(G-v)\circ\{v_2,v_5\}$; the first graph in the second bracket is $G_2=(G-v)\circ\{v_2,v_4\}$; and the first graph in the third bracket is $G_3=(G-v)\circ\{v_3,v_5\}$. The proof is completed.

3 Two Mathematical Ideas for Attacking Four-Color Conjecture Based on Theorem 2

It is well-known that mathematical induction is an effective method to prove Four-Color Conjecture, in which maximal planar graphs are classified into three cases by their minimum degrees. The case of minimum degree 3 or 4 is easy to prove by induction, but for the case of minimum degree 5 no mathematical method has been found. Based on Theorems 1 and 2, we give a new method to prove Four-Color Conjecture as follows.

In order to prove $f(G,4)>0$ for a maximal planar graph G, we use a mathematical inductive method on the number n of vertices of G.

When $n=3,4,5$, the result is obviously true.

Assume that $n\ge 5$ and the result is true for any maximal planar graph of order at most $n-1$. We consider the case that the order of graphs is n. We only consider simple maximal planar graphs. Notice that for any maximal planar graph G, $3\le\delta(G)\le 5$. So we need to consider the following three cases based on the minimum degree.

Case 1 $\delta(G)=3$

Let $v\in V(G)$, $d(v)=3$, and $G_1=G[\overline{N(v)}]$, $G_2=G-v$. Then we can obtain that $G_1\cap G_2=G[N(v)]\cong K_3$. Notice that $G_1=G[\overline{N(v)}]\cong K_4$, we can obtain the following result by Lemma 2:

$$f(G,t)=f(G_1\cup G_2,t)=\frac{f(G_1,t)\times f(G_2,t)}{f(K_3,t)}$$
$$=(t-3)f(G_2,t)$$

By the induction hypothesis, $f(G_2,4)>0$. Thus, $f(G,4)=f(G_2,4)>0$.

Hence, the result is true when $\delta(G)=3$.

Case 2 $\delta(G)=4$

Let $v\in V(G)$, $d(v)=4$, and $N(v)=\{v_1,v_2,v_3,v_4\}$ (see Fig. 2). Notice that we use $G[\overline{N(v)}]$ to denote G. By Theorem 1, we have $f(G,4)=f(G_1,4)+f(G_2,4)$, where $G_1=(G-v)\circ\{v_1,v_3\}$ and $G_2=(G-v)\circ\{v_2,v_4\}$. It is easy to prove that the graphs G_1 and G_2 are both maximal planar graphs with $n-2$ vertices. By the induction hypothesis, we can obtain

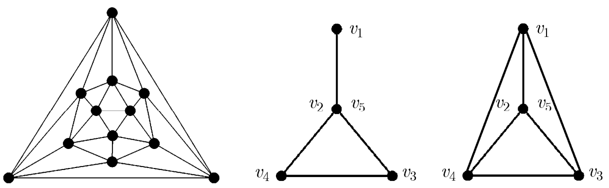
$$f(G_1,4)=f\left((G-v)\circ\{v_1,v_3\},4\right)>0$$
$$f(G_2,4)=f\left((G-v)\circ\{v_2,v_4\},4\right)>0$$

Therefore, $f(G,4)=f(G_1,4)+f(G_2,4)>0$ and the result is true when $\delta(G)=4$.

The key ingredient of the proof is the following Case 3.

Case 3 $\delta(G)=5$

The maximal planar graph of minimum degree 5 with fewest vertices is the icosahedron, depicted in Fig. 4(a), which has 12 vertices. Obviously, the icosahedron is 4-colorable. There is no maximal planar graph of minimum degree 5 with 13 vertices. Notice that for any maximal planar graph G of order at least 14 and minimum degree 5, there exists a vertex $v \in V(G)$ such that $d(v)=5$ and $d(v_1) \geq 6$, where $N(v)=\{v_1,v_2,v_3,v_4,v_5\}$ (see Fig. 3). Hence, the graph G_1 in Theorem 2 is a 4-colorable maximal planar graph of minimum degree at least 4. Based on this evidence, we give two mathematical ideas to prove $f(G,4)>0$ as follows.

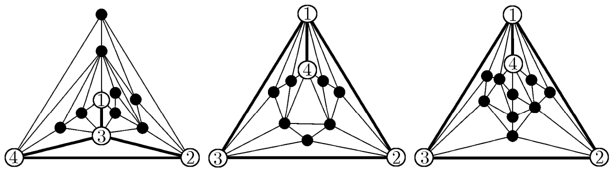


(a) Icosahedron　(b) Funnel subgraph　(c) K_4

Fig. 4　Three graphs in Case 3

The first idea is based on the fact that the value of each square bracket in Formula (9) is no less than zero. Hence, the Four-Color Conjecture can be proved if any square bracket's value is greater than zero. The value of the first square bracket is greater than zero if and only if there exists $f_1 \in C_4^0(G_1)$ such that $f_1(v_1)=f_1(v_3)$ or $f_1(v_1)=f_1(v_4)$. Therefore, $f(G,4)=0$ if and only if each square bracket in Formula (9) is equal to zero. Moreover, the value of the first square bracket is zero if and only if for any $f_1 \in C_4^0(G_1)$, $f_1(v_1) \neq f_1(v_3)$ and $f_1(v_1) \neq f_1(v_4)$, that is, for any $f_1 \in C_4^0(G_1)$, the colors of vertices of the funnel shown in Fig. 4(b) are pairwise different. Such maximal planar graphs are called 4-chromatic-funnel pseudo uniquely-4-colorable maximal planar graphs. For instance, each graph in Fig. 5 is a 4-chromatic-funnel pseudo-uniquely 4-colorable maximal planar graph.

A k-colorable graph G is called a k- colorable coordinated graph if there exist k vertices $v_1,v_2,\cdots,v_k$ in G such that $f(v_1)$, $f(v_2),\cdots$, $f(v_k)$ are pairwise different for any k-coloring



(a) 4-Chromatic-funnel type graph of order 12　(b) 4-Chromatic-funnel type graph of order 11　(c) 4-Chromatic-funnel type graph of order 13

Fig. 5 Three 4-chromatic-funnel pseudo uniquely-4-colorable maximal planar graphs

f of G. All 4-colorable coordinated maximal planar graphs can be divided into three classes: (1) uniquely 4-colorable maximal planar graphs, namely these graphs have only one partition of k -color class; (2) quasi uniquely-4- colorable maximal planar graphs, namely these graphs contain a subgraph that is uniquely 4- colorable; (3) pseudo uniquely-4-colorable maximal planar graphs, namely these graphs that are neither uniquely 4-colorable nor quasi uniquely-4-colorable. A detailed research on 4-colorable coordinated maximal planar graphs will be given in the subsequent series of articles.

Now we give the second idea to prove $f(G,4)>0$. The maximal planar graphs G_1, G_2, and G_3 in Theorem 2 can be regarded as the graphs obtained from G by deleting a 5-degree vertex v and contracting $\{v_2,v_5\}$, $\{v_2,v_4\}$, and $\{v_3,v_5\}$ into a single vertex, respectively (see Fig. 6). Moreover, the 5-cycle consisting of the neighbors of v in G is contracted to a funnel subgraph $L_1= v_1-\Delta v_2^5 v_3 v_4$ in G_1, $L_2=v_3-\Delta v_2^4 v_1 v_5$ in G_2, and $L_3=v_4-\Delta v_3^5 v_1 v_2$ in G_3, respectively, where v_2^5, v_2^4, and v_3^5 are the new vertices obtained by contracting $\{v_2,v_5\}$, $\{v_2,v_4\}$, and $\{v_3,v_5\}$, respectively.

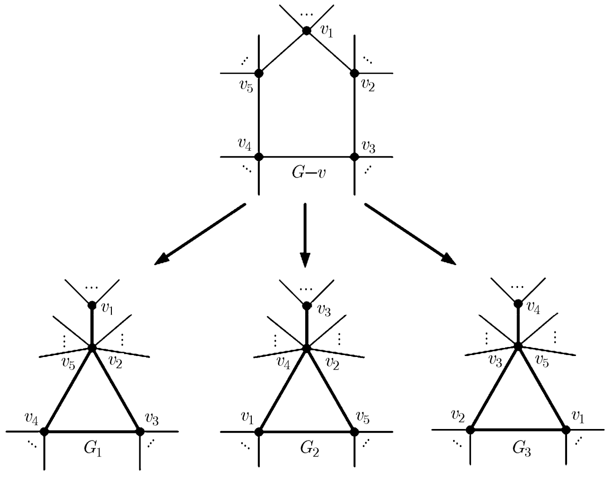


Fig. 6　The processes of generating the three funnel subgraphs

By the induction hypothesis, G_1, G_2, and G_3 are 4-colorable. In order to prove $f(G,4)>0$, it is needed to prove that at least one of the funnel subgraphs L_1, L_2, and L_3 is not 4-chromatic.

Therefore, the second idea is to prove that for any maximal planar graph G of minimum degree 5, there exists a 5-wheel W_5^v in G such that at least one of the funnel subgraphs L_1, L_2, and L_3 corresponding to G_1, G_2, and G_3 is not a 4-chromatic-funnel. For instance, the graph in Fig. 5(a) can be regarded as the maximal planar graph obtained from the graph in Fig. 7 by the operation shown in Fig. 6. It is not difficult to prove that the other two graphs obtained from Fig. 7 by the operation shown in Fig. 6 have no 4-chromatic-funnel.

4 Conclusion

In this paper, we give two recursion formulae of chromatic polynomial on maximal planar graphs. Based on these formulae, we find: (1) two mathematical ideas for attacking Four-Color Conjecture; (2) a method to generate maximal planar graphs, called contracting and extending operational system, which establishes a relation between the structure and colorings of a maximal planar graph. For instance, the maximal planar graph in Fig. 5(a) can be obtained from the graph in Fig. 7 by the extending 5-wheel operation, in other words, the maximal planar graph in Fig. 7 can be obtained from the graph in Fig. 5(a) by the contracting 5-wheel operation. A detailed research on contracting and extending operational system of maximal planar graphs will be given in later articles.

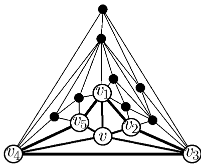


Fig. 7 A maximal planar graph that can be contracted to the graph in Fig. 5(a)

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